3 Regeneration of Neutral Kaons

Before diving into the (somewhat lengthy) formalism of neutral kaon regeneration, let us first elucidate the physical essence of this important phenomenon in simple terms. We emphasize that regeneration a priori has nothing to do with CP violation. Therefore this irrelevant complication will be ignored in the following introductory remarks.

Let us imagine the production of a neutral kaon beam, for instance by protons impinging on a target. If we further assume that the kaons leave the target at a momentum of a few GeV then the short-lived K_S component will have decayed after a few metres and we are left with a pure K_L beam. (Actually this separation of the two mass eigenstates is a great experimental boon!) Now we send this K_L beam through another block of matter, carbon for instance. The strongly interacting carbon nuclei "see" the incoming K_L as a superposition of the strangeness states K^0 and $\overline{K}^0$. These two components will interact very differently in matter: $\overline{K}^0$ mesons are effectively absorbed through hyperon production processes such as

$$\overline{K}^0 + p \to \Lambda + \pi^+$$

whereas K^0 mesons can only scatter elastically or undergo charge exchange. Let f and $\overline{f}$ denote the amplitudes of K^0 and $\overline{K}^0$ scattering off a nucleus. The scattering will transform the $K_L = \frac{1}{\sqrt{2}}(K^0 + \overline{K}^0)$ state into

$$\frac{1}{\sqrt{2}}(f K^0 + \overline{f}\,\overline{K}^0) = \frac{1}{2} f(K_L + K_S) + \frac{1}{2}\overline{f}(K_L - K_S) = \frac{1}{2}(f + \overline{f})K_L + \frac{1}{2}(f - \overline{f})K_S.$$

Hence, a K_S component is *regenerated* with an amplitude proportional to $f - \overline{f}$.

This at first baffling result is quite comprehensible if K^0 and $\overline{K}^0$ are envisaged as orthogonal basis vectors in a two-dimensional plane in which K_L and K_S form an equivalent basis, rotated by an angle of 45°. The decay of the K_S component corresponds to a projection of the state onto the K_L axis. Inside the absorbing material, the $\overline{K}^0$ component (or a large fraction of it) is projected out of the K_L state, leaving behind a state with predominant K^0 content. This state must contain a K_S component. Simple analogies involving linearly polarized light or atoms in a Stern–Gerlach type experiment can be found in various textbooks [24, 25].

Although the name *regeneration* clearly comes from the reappearance of K_S decays from a pure K_L beam outlined above, we will use it as a generic term to describe any transition between the mass eigenstates. Because of the non-zero mass difference, such transitions can only occur in the presence of matter (in contrast to strangeness oscillations).

The regeneration phenomenon was predicted by Pais and Piccioni [26] in 1955, shortly after Gell-Mann and Pais had published their particle mixture hypothesis [3], and still before the discovery of the K_L. In the following years the theoretical aspects of the "Pais–Piccioni experiment," as regeneration was called in those days, were thoroughly analysed by Case [27] and Good [28]. The experimental search for regeneration succeeded in 1961 when R.H. Good and co-workers reported the first observation of K_S decays behind iron and lead plates exposed in a K_L beam [29].

3.1 Types of regeneration

According to the different ways of kaon scattering in condensed matter one generally distinguishes three classes of regeneration [30, 31]:

Coherent (transmission) regeneration: elastic scattering on nuclei in the forward direction where the scatterers act coherently over an extended region of several centimetres length.

Incoherent (diffraction) regeneration: elastic diffractive scattering off individual nuclei in any direction (including forward). Incoherent addition of intensities from different nuclei, but coherent action of the nucleons inside the nucleus.

Inelastic regeneration: inelastic scattering with large momentum transfer. Excitation or disintegration of the nucleus.

All three types of regeneration were observed, for the first time by R.H. Good et al. [29].

CPLEAR, being an interference experiment, is most susceptible to coherent regeneration. Only coherently regenerated kaons can produce additional interference terms and significantly disturb the measurement. The main effect of the incoherently regenerated kaons is to produce some additional two pion decays which may be treated as a further source of (asymmetric) background. It is in most cases negligible. Inelastic regeneration is completely negligible at energies available to CPLEAR as the corresponding cross-sections are one or two orders of magnitude smaller than their elastic counterparts.

In the following Subsection we present in detail the phenomenology of coherent regeneration *for arbitrary mixtures of* K^0 *and* $\overline{K}^0$ *states.*[3] Ever since the pioneering work of Case [27] and Good [28] this topic has not received much attention because most experiments involving neutral kaon regeneration send a pure K_L beam through a regenerator. For these measurements it is sufficient to know the K_S amplitude after the regenerator. For experiments employing pure K^0 and $\overline{K}^0$ beams like CPLEAR or detectors at ϕ-factories, however, a K_L–K_S symmetric approach taking into account all possible interference terms is imperative. In Section 3.3 we will turn to the effects of incoherent regeneration, again working with arbitrary mixtures of K^0 and $\overline{K}^0$.

3.2 Coherent (transmission) regeneration

Coherent regeneration is a further example of how the tiny mass difference $\Delta m = m_L - m_S$ between K_L and K_S gives rise to phenomena involving macroscopic distances, in this case a coherence length of several centimetres. To see this[4] let us consider the momentum transfer in a forward scattering reaction of the type K_L + nucleus $\to K_S$ + nucleus. The incoming K_L has energy and momentum (four-momentum) $(E_L, \mathbf{p}_L)$ before interacting with a nucleus of mass M at rest. After the scattering we have an outgoing K_S with $(E_S, \mathbf{p}_S)$ and a nucleus carrying the transferred energy and momentum,

$$\begin{pmatrix} E' \\ \mathbf{p}' \end{pmatrix} = \begin{pmatrix} M + E_L - E_S \\ \mathbf{p}_L - \mathbf{p}_S \end{pmatrix}.$$

Energy and momentum conservation further require

$$\begin{aligned}(E_L - E_S)^2 - (p_L^2 - p_S^2)^2 &= (E' - M)^2 - p'^2 \\ &= 2M^2 - 2ME' \\ &= 2M(E_L - E_S).\end{aligned}$$

Solving for the energy difference $(E_L - E_S)$ we find that it vanishes provided that $M^2 \gg (p_L - p_S)^2$. Considering the minute momentum transfer needed to turn a K_L into a K_S without changing the direction of flight we may certainly consider this assumption to be valid. Thus we have $E_L = E_S$ and may write

$$p_L^2 + m_L^2 = p_S^2 + m_S^2$$

from which we deduce

$$p_S - p_L = \Delta m \frac{m_{K^0}}{p_{K^0}} \tag{3.1}$$

where the K^0-subscript denotes the mean mass and momentum. Now imagine a second scatterer, some distance d further down the neutral kaon flight path. Our K_L may as well choose to regenerate into a K_S at this nucleus instead of the first, bringing about a second K_S amplitude. Will there be interference between the two amplitudes (i.e. will the two scatterers act coherently)? The amplitude regenerated at the first nucleus will have acquired an additional phase $p_S d$ when arriving at the second nucleus. The amplitude to be regenerated at the second nucleus travels the distance d as a K_L and therefore its phase will change by $p_L d$. (Both phases are equally affected by the regeneration process itself.) The K_S intensity will be proportional to

$$|e^{ip_S d} + e^{ip_L d}|^2 = 2 + 2\cos((p_S - p_L)d),$$

3. Part of this work has been published in Ref. 32.
4. Our derivation of the coherence conditions essentially follows Ref. 30.

so the coherence condition reads

$$(p_S - p_L)d \lesssim 1. \tag{3.2}$$

Inserting (3.1) we see that for low kaon momenta ($p_{K^0} \approx m_{K^0}$) the maximum separation of two nuclei still acting coherently on the neutral kaon is given by the inverse of the mass difference Δm:

$$d_{\text{max}} \approx \frac{1}{\Delta m} \approx 6 \text{ cm}, \tag{3.3}$$

and even larger at higher momenta. This result reflects the well known (and much lamented) fact that the resolution achievable in a scattering experiment is inversely proportional to the momentum transfer occurring in the reaction, usually denoted by q: we build stronger and stronger accelerators to study matter at smaller and smaller distances. Here, however, we find ourselves at the very opposite end of the spectrum: the momentum transfer needed to change a K_L into a K_S is so small, of the order of $\Delta m \approx 3.5\ \mu$eV, that in this (very peculiar) scattering experiment we do not resolve structures smaller than several centimetres! Over such distances we cannot tell where the scattering took place. This means that a neutral kaon undergoing coherent regeneration in a solid plate of a few centimetres thickness interacts with the plate *as a whole*.[5] In a formal treatment of coherent regeneration it is therefore most appropriate to describe the scatterer by a macroscopic variable, namely an *index of refraction* (see the following Subsection).

The maximum scattering angle θ_{max} at which coherent regeneration is still possible is also found by considering the phase difference of the K_S amplitudes arising from two scattering centres. If the K_S is emitted under an angle θ with respect to the K_L beam axis then the phase difference in (3.2) decreases to

$$(p_S \cos\theta - p_L)d \approx \left[(p_S - p_L) - p_S\frac{\theta^2}{2}\right] d.$$

The additional term due to the finite scattering angle θ must be smaller than the original phase shift in order not to significantly disturb the coherence condition (3.2):

$$p_S\frac{\theta^2}{2} \lesssim p_S - p_L$$

This gives

$$\theta^2_{\text{max}} \approx 2\frac{p_S - p_L}{p_{K^0}}, \tag{3.4}$$

or $\theta_{\text{max}} \approx 10^{-7}$ for low momenta and less for higher momenta.

3.2.1 Coherent scattering and complex index of refraction

We treat the regenerating medium as a distribution of scatterers. An incident coherent wave of wave number (momentum) k will have a different wave number k' inside the medium because of the interaction with the scatterers. For randomly distributed scatterers the difference between the incident coherent field and the effective field inside the medium may be neglected and the relation between the wave numbers is given by [33]

$$(k')^2 = k^2 + 4\pi N f_{k'}(0) \tag{3.5}$$

where N is the density of scattering centres and $f_{k'}(0)$ is the elastic forward scattering amplitude evaluated at k'. Assuming that the additional term in (3.5) is very small we can calculate the index of refraction of the medium as

$$n = \frac{k'}{k} = 1 + \frac{2\pi N}{k^2} f_k(0). \tag{3.6}$$

It should be kept in mind that the above formula rests on two assumptions: firstly the scattering centres (i.e. the nuclei) are distributed completely at random and secondly $|n - 1| \ll 1$ so that we may set $f_{k'}(0) = f_k(0)$ and hence linearize in $(n - 1)$.

5. Coherent regeneration is sometimes called "regeneration by a plate" [29].

3.2.2 *The effective Hamiltonian in matter and the regeneration parameter*

Here we carry on our investigation of neutral kaon dynamics governed by an effective Hamiltonian from Section 2.1 and will extend it to incorporate the effects of a strongly interacting medium. For this phenomenological analysis of regeneration we assume CPT invariance but not T invariance, so that *in vacuum* we are dealing with an effective Hamiltonian of the form

$$\mathbf{H} = D\mathbb{1} + E_1\sigma_1 + E_2\sigma_2 \tag{3.7}$$

and its eigenstates

$$\mathrm{K_L} = \frac{1}{\sqrt{2(1+|\epsilon_\mathrm{T}|^2)}}[(1+\epsilon_\mathrm{T})\mathrm{K}^0 + (1-\epsilon_\mathrm{T})\overline{\mathrm{K}}^0], \tag{3.8a}$$

$$\mathrm{K_S} = \frac{1}{\sqrt{2(1+|\epsilon_\mathrm{T}|^2)}}[(1+\epsilon_\mathrm{T})\mathrm{K}^0 - (1-\epsilon_\mathrm{T})\overline{\mathrm{K}}^0], \tag{3.8b}$$

which we will call mass eigenstates. A general state ψ can be written as a linear combination of any pair of eigenstates:

$$\psi(\tau) = \alpha(\tau)\mathrm{K}^0 + \overline{\alpha}(\tau)\overline{\mathrm{K}}^0 = \alpha_\mathrm{L}(\tau)\mathrm{K_L} + \alpha_\mathrm{S}(\tau)\mathrm{K_S} \tag{3.9}$$

$\alpha_\mathrm{L}(\tau)$ and $\alpha_\mathrm{S}(\tau)$ are the amplitudes for finding the state as a $\mathrm{K_L}$ or a $\mathrm{K_S}$, respectively. From (3.8) we get the following relations for the amplitudes:

$$\alpha = \frac{1+\epsilon}{\sqrt{1+|\epsilon|^2}}\frac{\alpha_\mathrm{L}+\alpha_\mathrm{S}}{\sqrt{2}} \tag{3.10a}$$

$$\overline{\alpha} = \frac{1-\epsilon}{\sqrt{1+|\epsilon|^2}}\frac{\alpha_\mathrm{L}-\alpha_\mathrm{S}}{\sqrt{2}} \tag{3.10b}$$

Now to accommodate for the strong interaction of the kaons with the nuclei of the surrounding medium we have to add to the effective Hamiltonian (3.7) a nuclear term that will be diagonal in the K^0–$\overline{\mathrm{K}}^0$ basis:

$$\mathbf{H}' = \mathbf{H} + \mathbf{H}_\mathrm{nuc} \tag{3.11}$$

The nuclear contribution to the rate of change of a general state ψ (3.9) is given by the indices of refraction for K^0 and $\overline{\mathrm{K}}^0$, defined in the last Paragraph:

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)_\mathrm{nuc} = \mathrm{i}k\begin{pmatrix} n-1 & 0 \\ 0 & \overline{n}-1 \end{pmatrix}\psi$$

We transform into the particle's rest frame and insert (3.6):

$$\mathrm{i}\left(\frac{\mathrm{d}\psi}{\mathrm{d}\tau}\right)_\mathrm{nuc} = -\frac{2\pi N}{m}\begin{pmatrix} f & 0 \\ 0 & \overline{f} \end{pmatrix}\psi \tag{3.12}$$

Comparing with Schrödinger's equation for the time evolution we find

$$\mathbf{H}_\mathrm{nuc} = \begin{pmatrix} \varkappa & 0 \\ 0 & \overline{\varkappa} \end{pmatrix} \tag{3.13}$$

with

$$\varkappa = -\frac{2\pi N}{m}f \quad \text{and} \quad \overline{\varkappa} = -\frac{2\pi N}{m}\overline{f}, \tag{3.14}$$

defined in analogy to the eigenvalues of the free Hamiltonian λ_L and λ_S. The interaction with matter leads to a term proportional to σ_3 in the effective Hamiltonian, mimicking a violation of CPT. (This comes as no surprise as our regenerating medium consists solely of matter and not antimatter.) To obtain the eigenvalues and eigenvectors of $\mathbf{H}'$ we can therefore simply apply the general formulae from Section 2.1. Writing

$$\mathbf{H}' = D'\mathbb{1} + \mathbf{E}' \cdot \sigma \tag{3.15}$$

we find that

$$D' = D + \frac{\varkappa + \overline{\varkappa}}{2}, \quad E_3' = \frac{\varkappa - \overline{\varkappa}}{2}. \tag{3.16}$$

Our expression for the index of refraction (3.6) is linearized in $\varkappa$. Consequently, we must drop terms proportional to E_3^2 and set $E' = E$. The eigenvalues of the new effective Hamiltonian are then

$$\lambda_{\mathrm{L,S}}' = D' \pm E = \lambda_{\mathrm{L,S}} + \frac{\varkappa + \overline{\varkappa}}{2}. \tag{3.17}$$

For the eigenstates we employ expressions (2.15) with δ_{CPT} replaced by

$$\delta' = \frac{E_3'}{E + E_1}. \tag{3.18}$$

δ' is, to first order in ϵ_{T}, equal to the *regeneration parameter* r defined as

$$r := \frac{\Delta\varkappa}{2\Delta\lambda} = -\frac{\pi N}{m} \frac{\Delta f}{\Delta m - (\mathrm{i}/2)\Delta\Gamma}. \tag{3.19}$$

(In all our equations $\Delta x \equiv x_{\mathrm{L}} - x_{\mathrm{S}}$ for quantities referring to the mass eigenstates and $\Delta y \equiv y - \overline{y}$ for variables associated with strangeness.) The parameter r is sometimes expressed in terms of the mean life τ_{S} and the mean decay length $\Lambda_{\mathrm{S}} = \gamma v \tau_{\mathrm{S}}$ of the $\mathrm{K_S}$ as

$$r = \mathrm{i}\pi \frac{f - \overline{f}}{p} \frac{N\Lambda_{\mathrm{S}}}{\frac{1}{2} - \mathrm{i}\Delta m \tau_{\mathrm{S}}} \tag{3.20}$$

where $p = \gamma v m$ is the particle's momentum, v its velocity relative to the regenerator, $\gamma = (1 - v^2)^{-1/2}$ the usual Lorentz factor and Γ_{L} is neglected with respect to Γ_{S} [34]. The magnitude of r is typically of the order of 10^{-2} for condensed matter, i.e. regeneration effects are fairly small in most materials. The eigenstates in matter may now be written as

$$\mathrm{K_L'} = \frac{1}{\sqrt{2(1 + |\epsilon_{\mathrm{T}} + r|^2)}}[(1 + \epsilon_{\mathrm{T}} + r)\mathrm{K}^0 + (1 - \epsilon_{\mathrm{T}} - r)\overline{\mathrm{K}}^0], \tag{3.21a}$$

$$\mathrm{K_S'} = \frac{1}{\sqrt{2(1 + |\epsilon_{\mathrm{T}} + r|^2)}}[(1 + \epsilon_{\mathrm{T}} - r)\mathrm{K}^0 - (1 - \epsilon_{\mathrm{T}} + r)\overline{\mathrm{K}}^0]. \tag{3.21b}$$

Neglecting furthermore quadratic terms in r we arrive at

$$\begin{aligned} \mathrm{K_L'} &= \frac{1}{\sqrt{2}}[(1 + \epsilon_{\mathrm{T}})\mathrm{K}^0 + (1 - \epsilon_{\mathrm{T}})\overline{\mathrm{K}}^0 + r(\mathrm{K}^0 - \overline{\mathrm{K}}^0)] \\ &= \mathrm{K_L} + r\mathrm{K_S}, \end{aligned} \tag{3.22a}$$

$$\mathrm{K_S'} = \mathrm{K_S} - r\mathrm{K_L}. \tag{3.22b}$$

Having found the eigenstates and eigenvalues of the effective Hamiltonian, the time development in matter is readily computed for a general state

$$\psi(\tau) = \alpha_{\mathrm{L}}(\tau)\mathrm{K_L} + \alpha_{\mathrm{S}}(\tau)\mathrm{K_S} = \alpha_{\mathrm{L}}'(\tau)\mathrm{K_L'} + \alpha_{\mathrm{S}}'(\tau)\mathrm{K_S'}.$$

Using (3.22) and (3.17) we have (always to first order in ϵ_{T} and r)

$$\begin{aligned} \alpha_{\mathrm{L}}(\tau) &= \alpha_{\mathrm{L}}'(0)\mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{L}}'\tau} - r\alpha_{\mathrm{S}}'(0)\mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{S}}'\tau} \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2}(\varkappa + \overline{\varkappa})\tau}\left[\alpha_{\mathrm{L}}(0)\mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{L}}\tau} + r\alpha_{\mathrm{S}}(0)\left(\mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{L}}\tau} - \mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{S}}\tau}\right)\right] \\ &= \mathrm{e}^{-\frac{\mathrm{i}}{2}(\varkappa + \overline{\varkappa})\tau}\mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{L}}\tau}(\alpha_{\mathrm{L}}(0) + \varrho\alpha_{\mathrm{S}}(0)) \end{aligned} \tag{3.23a}$$

and similarly

$$\alpha_S(\tau) = e^{-\frac{i}{2}(\varkappa+\overline{\varkappa})\tau} e^{-i\lambda_S\tau} \left(\alpha_S(0) + \varrho\alpha_L(0)e^{-i\Delta\lambda\tau}\right), \tag{3.23b}$$

where we define the *geometry-dependent regeneration parameter* $\varrho(L)$ as the fraction of K_S in an initially pure K_L beam after penetrating a regenerator of thickness $L = \gamma v \tau$:

$$\varrho(L) \equiv \frac{\alpha_S(L)}{\alpha_L(L)} \quad \text{for} \quad \alpha_L(0) \doteq 1 \quad \text{and} \quad \alpha_S(0) = 0. \tag{3.24}$$

This ratio is—to first order in r—equal to r times a geometrical factor in accordance with (3.23) (see also Eq. 3.33 in the following Paragraph):

$$\varrho(L) = r\left[1 - \exp\left(i\Delta\lambda\frac{L}{\gamma v}\right)\right] \tag{3.25}$$

In the notation of Eq. 3.20 this reads [30]

$$\varrho(L) = i\pi\frac{f-\overline{f}}{p}N\Lambda_S\frac{1-\exp\left[\left(i\Delta m\tau_S - \frac{1}{2}\right)\frac{L}{\Lambda_S}\right]}{\frac{1}{2} - i\Delta m\tau_S}. \tag{3.26}$$

3.2.3 *Time development of a general state in matter*

The simple formulae for the time development of a general state in matter obtained in the last Paragraph only hold in first order of ϵ_T and r. To derive exact expressions it is more convenient to directly solve the Schrödinger equation

$$i\frac{d\psi}{d\tau} = \mathbf{H}'\psi = (\mathbf{H} + \mathbf{H}_{nuc})\psi \tag{3.27}$$

as was first done by Case [27] and Good [28]. The eigenvalues and eigenvectors of either part of the Hamiltonian are well known and their respective contributions to the change of the state ψ with respect to proper time τ are given by

$$i\left(\frac{d\psi}{d\tau}\right)_{vac} = \lambda_L\alpha_L K_L + \lambda_S\alpha_S K_S, \tag{3.28}$$

$$i\left(\frac{d\psi}{d\tau}\right)_{nuc} = \varkappa\alpha K^0 + \overline{\varkappa}\overline{\alpha}\overline{K}^0. \tag{3.29}$$

Applying (3.10) we express the sum in the basis of the mass eigenstates and obtain Good's equations in the particle's rest frame:

$$i\frac{d\alpha_L}{d\tau} = \lambda_L\alpha_L + \left(\frac{\varkappa+\overline{\varkappa}}{2}\right)\alpha_L + \left(\frac{\varkappa-\overline{\varkappa}}{2}\right)\alpha_S \tag{3.30a}$$

$$i\frac{d\alpha_S}{d\tau} = \lambda_S\alpha_S + \left(\frac{\varkappa+\overline{\varkappa}}{2}\right)\alpha_S + \left(\frac{\varkappa-\overline{\varkappa}}{2}\right)\alpha_L \tag{3.30b}$$

CP violation does not manifest itself in the equations, a fact pointed out by Good already. The solution of (3.30) may be written as[6]

$$\alpha_L(\tau) = e^{-i\Sigma\cdot\tau}\left[\alpha_L^0\cos(\Omega\tau) - \frac{i}{2\Omega}(\Delta\lambda\alpha_L^0 + \Delta\varkappa\alpha_S^0)\sin(\Omega\tau)\right], \tag{3.31a}$$

$$\alpha_S(\tau) = e^{-i\Sigma\cdot\tau}\left[\alpha_S^0\cos(\Omega\tau) + \frac{i}{2\Omega}(\Delta\lambda\alpha_S^0 - \Delta\varkappa\alpha_L^0)\sin(\Omega\tau)\right], \tag{3.31b}$$

6. The general solution of the system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad \Leftrightarrow \quad \begin{aligned} \dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) \\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) \end{aligned}$$

with constants

$$\Omega = \frac{1}{2}\sqrt{\Delta\lambda^2 + \Delta\varkappa^2}, \qquad \Sigma = \frac{1}{2}(\lambda_L + \lambda_S + \varkappa + \overline{\varkappa}).$$

α_L^0 and α_S^0 are the initial amplitudes for $\tau = 0$. The factor $e^{-i\Sigma\cdot\tau}$ appearing in both amplitudes describes the decay of the mass eigenstates as well as the absorption of the strangeness eigenstates:

$$\left|e^{-i\Sigma\cdot\tau}\right|^2 = e^{-\frac{1}{2}(\Gamma_L+\Gamma_S)\tau} e^{-\frac{N}{2}(\sigma_T+\overline{\sigma}_T)\gamma v\tau}$$

where σ_T and $\overline{\sigma}_T$ are the total cross sections for K^0 and $\overline{K}^0$, respectively, and we have used the optical theorem. In terms of the regeneration parameter[7] r defined in the preceding Paragraph, (3.31) reads

$$\alpha_L(\tau) = e^{-i\Sigma\cdot\tau}\left[\alpha_L^0 \cos(\frac{\Delta\lambda}{2}\sqrt{1+4r^2}\tau) - i\frac{\alpha_L^0 + 2r\alpha_S^0}{\sqrt{1+4r^2}}\sin(\frac{\Delta\lambda}{2}\sqrt{1+4r^2}\tau)\right], \quad (3.32a)$$

$$\alpha_S(\tau) = e^{-i\Sigma\cdot\tau}\left[\alpha_S^0 \cos(\frac{\Delta\lambda}{2}\sqrt{1+4r^2}\tau) + i\frac{\alpha_S^0 - 2r\alpha_L^0}{\sqrt{1+4r^2}}\sin(\frac{\Delta\lambda}{2}\sqrt{1+4r^2}\tau)\right]. \quad (3.32b)$$

In (3.31) and (3.32) quadratic terms in $\varkappa$, $\overline{\varkappa}$ and $\Delta\lambda/\lambda$ are neglected but no approximations concerning the smallness of ϵ_T and r have been made so far. For the calculation of the following Paragraph, however, we do take advantage of the fact that $r \ll 1$ and limit the discussion to first or second order processes. Expanding (3.32) up to second order in r we get

$$\alpha_L(\tau) = e^{-\frac{i}{2}(\varkappa+\overline{\varkappa})\tau}\left\{\alpha_L^0 e^{-i\lambda_L\tau} + r\alpha_S^0\left[e^{-i\lambda_L\tau} - e^{-i\lambda_S\tau}\right]\right.$$
$$\left. -r^2\alpha_L^0\left[e^{-i\lambda_L\tau} - e^{-i\lambda_S\tau} + i\Delta\lambda\tau e^{-i\lambda_L\tau}\right]\right\}, \quad (3.33a)$$

$$\alpha_S(\tau) = e^{-\frac{i}{2}(\varkappa+\overline{\varkappa})\tau}\left\{\alpha_S^0 e^{-i\lambda_S\tau} + r\alpha_L^0\left[e^{-i\lambda_L\tau} - e^{-i\lambda_S\tau}\right]\right.$$
$$\left. +r^2\alpha_S^0\left[e^{-i\lambda_L\tau} - e^{-i\lambda_S\tau} + i\Delta\lambda\tau e^{-i\lambda_S\tau}\right]\right\}. \quad (3.33b)$$

It is easily verified that the first order approximation reduces to (3.23).

3.2.4 Two pion decay after penetration of a regenerator

Let us consider a neutral kaon which, at $\tau = 0$, is in an arbitrary state and approaches a regenerator of thickness L with relative velocity v. It enters the regenerator at proper time $\tau_1 \geqslant 0$ and leaves it again at

$$\tau_2 = \tau_1 + L/\gamma v \equiv \tau_1 + \delta\tau$$

(see Fig. 1). Before entering the regenerator the two mass components evolve separately according to

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is given by

$$x_1(t) = C_{11}e^{\lambda_1 t} + C_{12}e^{\lambda_2 t},$$
$$x_2(t) = C_{21}e^{\lambda_1 t} + C_{22}e^{\lambda_2 t},$$

where λ_1 and λ_2 are the eigenvalues of $\mathbf{A}$, $\lambda_{1,2} = a^+ \pm \Delta$ with $a^\pm := \frac{1}{2}(a_{11} \pm a_{22})$ and $\Delta := \sqrt{(a^-)^2 + a_{12}a_{21}}$. We express the solution in the basis of the hyperbolic functions,

$$x_1(t) = e^{a^+ t}[C_c \cosh(\Delta\cdot t) + C_s \sinh(\Delta\cdot t)],$$
$$x_2(t) = e^{a^+ t}[C_c' \cosh(\Delta\cdot t) + C_s' \sinh(\Delta\cdot t)],$$

and determine the constants from the initial values at $t = 0$:

$$C_c = x_1(0) \qquad C_s = \Delta^{-1}[a^- x_1(0) + a_{12}x_2(0)]$$
$$C_c' = x_2(0) \qquad C_s' = \Delta^{-1}[a_{21}x_1(0) - a^- x_2(0)]$$

7. Our parameter r is essentially equal to the coefficient $R/(1 - R^2)$ in Good's solution of the differential equations.

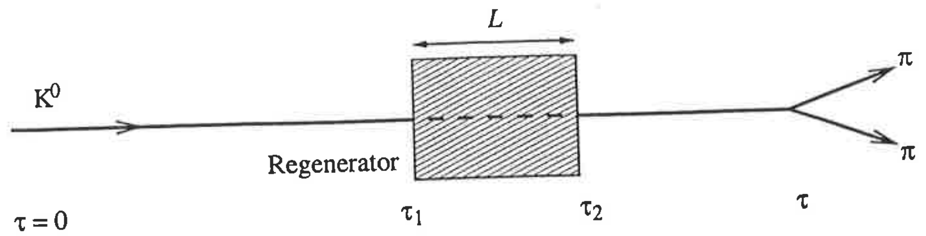

Figure 1: Neutral kaon traversing a regenerator before decaying into two pions.

(3.28):

$$\alpha_{L,S}(\tau_1) = e^{-i\lambda_{L,S}\tau_1}\alpha^0_{L,S},$$

whereas inside the regenerator the components mix as described by (3.33):

$$\alpha_{L,S}(\tau_2) = e^{-\frac{i}{2}(\varkappa+\overline{\varkappa})\tau}\Big\{\alpha_{L,S}(\tau_1)e^{-i\lambda_{L,S}\delta\tau} + r\alpha_{S,L}(\tau_1)\left[e^{-i\lambda_L\delta\tau} - e^{-i\lambda_S\delta\tau}\right] \mp r^2\alpha_{L,S}(\tau_1)\left[e^{-i\lambda_L\delta\tau} - e^{-i\lambda_S\delta\tau} + i\Delta\lambda\delta\tau e^{-i\lambda_{L,S}\delta\tau}\right]\Big\}.$$

Afterwards the wave propagates through vacuum again, so that for life times $\tau \geqslant \tau_2$ we have

$$\alpha_{L,S}(\tau) = e^{-i\lambda_{L,S}(\tau-\tau_2)}\alpha_{L,S}(\tau_2).$$

The two pion decay rate of a general state (3.9) can be reduced to the corresponding time-independent decay rate R_S of the K_S eigenstate:

$$R(\tau) = R_S\,|\alpha_S(\tau) + \eta\alpha_L(\tau)|^2 \tag{3.34}$$

where η is the ratio of the CP violating to the CP conserving decay amplitude as defined in (2.25–2.27). Inserting the above recursive formulae for the amplitudes into (3.34) we arrive at

$$R(\tau) = R_S\,e^{-\frac{N}{2}(\sigma_T+\overline{\sigma}_T)L}\left(e^{-\Gamma_S\tau}\mathcal{S} + 2|\eta|e^{-\frac{1}{2}(\Gamma_L+\Gamma_S)\tau}\mathcal{M} + |\eta|^2e^{-\Gamma_L\tau}\mathcal{L}\right), \tag{3.35}$$

with

$$\begin{aligned}\mathcal{L},\mathcal{S} = {} & |\alpha^0_{L,S}|^2 \\ & \pm 2|r||\alpha^0_L||\alpha^0_S|\Big[e^{\pm\frac{1}{2}\Delta\Gamma\tau_1}\cos(\Delta m\tau_1 \pm \varphi_r - \Delta\varphi^0) \\ & \qquad - e^{\pm\frac{1}{2}\Delta\Gamma\tau_2}\cos(\Delta m\tau_2 \pm \varphi_r - \Delta\varphi^0)\Big] \\ & + |r|^2|\alpha^0_{S,L}|^2\Big[e^{\pm\Delta\Gamma\tau_1} + e^{\pm\Delta\Gamma\tau_2} - 2e^{\pm\frac{1}{2}\Delta\Gamma(\tau_1+\tau_2)}\cos(\Delta m\delta\tau)\Big] \\ & + 2|r|^2|\alpha^0_{L,S}|^2\Big[e^{\pm\frac{1}{2}\Delta\Gamma\delta\tau}\cos(\Delta m\delta\tau \pm 2\varphi_r) \\ & \qquad - (1 \pm \frac{1}{2}\Delta\Gamma\delta\tau)\cos(2\varphi_r) \pm \Delta m\delta\tau\sin(2\varphi_r)\Big]\end{aligned}$$

and

$$\mathcal{M} = |\alpha^0_L||\alpha^0_S|\cos(\Delta m\tau - \phi - \Delta\varphi^0) + \mathcal{M}_1 + \mathcal{M}_2,$$

$$
\begin{aligned}
\mathcal{M}_{1,2} = \pm\, |r||\alpha^0_{\mathrm{S,L}}|^2 \Big[& \mathrm{e}^{\pm\frac{1}{2}\Delta\Gamma\tau_1}\cos(\Delta m(\tau-\tau_1)-\phi\mp\varphi_r) \\
& - \mathrm{e}^{\pm\frac{1}{2}\Delta\Gamma\tau_2}\cos(\Delta m(\tau-\tau_2)-\phi\mp\varphi_r)\Big] \\
+ |r|^2|\alpha^0_{\mathrm{L}}||\alpha^0_{\mathrm{S}}| \Big[& \mathrm{e}^{\pm\frac{1}{2}\Delta\Gamma\delta\tau}\cos(\Delta m(\tau-\tau_1-\tau_2)-\phi+\Delta\varphi^0) \\
& - \cos(\Delta m(\tau-2\tau_{1,2})-\phi+\Delta\varphi^0) \\
& + \mathrm{e}^{\pm\frac{1}{2}\Delta\Gamma\delta\tau}\cos(\Delta m(\tau-\delta\tau)-\phi\mp 2\varphi_r-\Delta\varphi^0) \\
& - (1\pm\frac{1}{2}\Delta\Gamma\delta\tau)\cos(\Delta m\tau-\phi\mp 2\varphi_r-\Delta\varphi^0) \\
& - \Delta m\delta\tau\sin(\Delta m\tau-\phi\mp 2\varphi_r-\Delta\varphi^0)\Big],
\end{aligned}
$$

where φ_r is the phase of r and $\Delta\varphi^0$ is defined as

$$
\Delta\varphi^0 = \varphi^0_{\mathrm{L}} - \varphi^0_{\mathrm{S}} = \arg(\alpha^0_{\mathrm{L}}) - \arg(\alpha^0_{\mathrm{S}}).
$$

The general formulae can easily be adapted to the physically relevant cases of initial eigenstates of definite strangeness as produced in strong interactions. For K^0 and $\overline{\mathrm{K}}^0$ as initial states we have from (3.8)

$$
|\alpha^0_{\mathrm{L}}|^2 = |\alpha^0_{\mathrm{S}}|^2 = |\alpha^0_{\mathrm{L}}||\alpha^0_{\mathrm{S}}| \approx \begin{cases} \frac{1}{2} - \mathrm{Re}\,\epsilon_{\mathrm{T}} & \text{for } \mathrm{K}^0 \\ \frac{1}{2} + \mathrm{Re}\,\epsilon_{\mathrm{T}} & \text{for } \overline{\mathrm{K}}^0 \end{cases}
$$

and

$$
\Delta\varphi^0 = \begin{cases} 0 & \text{for } \mathrm{K}^0 \\ \pi & \text{for } \overline{\mathrm{K}}^0, \end{cases}
$$

i.e. for an initial $\overline{\mathrm{K}}^0$ the terms containing $\Delta\varphi^0$ in the cosine argument change their signs. Hence the measurement of the time-dependent decay rate (3.35) for initial K^0 and $\overline{\mathrm{K}}^0$ allows the determination of the magnitude and the phase of the regeneration parameter r and so, via (3.19), of the regeneration amplitude Δf.

The terms of lowest order in r may be rewritten by use of the geometry-dependent regeneration parameter $\varrho(L=\gamma v\,\delta\tau)$, yielding

$$
\mathcal{L},\mathcal{S} = |\alpha^0_{\mathrm{L,S}}|^2 + 2|\varrho||\alpha^0_{\mathrm{L}}||\alpha^0_{\mathrm{S}}|\mathrm{e}^{\pm\frac{1}{2}\Delta\Gamma\tau_{1,2}}\cos(\Delta m\tau_{1,2}\pm\varphi_\varrho-\Delta\varphi^0)
$$

and

$$
\begin{aligned}
\mathcal{M} = & |\alpha^0_{\mathrm{L}}||\alpha^0_{\mathrm{S}}|\cos(\Delta m\tau-\phi-\Delta\varphi^0) \\
& + |\varrho||\alpha^0_{\mathrm{S}}|^2\,\mathrm{e}^{\frac{1}{2}\Delta\Gamma\tau_1}\cos(\Delta m(\tau-\tau_1)-\phi-\varphi_\varrho) \\
& + |\varrho||\alpha^0_{\mathrm{L}}|^2\,\mathrm{e}^{-\frac{1}{2}\Delta\Gamma\tau_2}\cos(\Delta m(\tau-\tau_2)-\phi+\varphi_\varrho)
\end{aligned}
$$

as lowest order approximations.

It is noteworthy that the decay rates for initial K^0 or $\overline{\mathrm{K}}^0$ exhibit a substantial *linear* dependence on the regeneration parameter r whereas for initial $\mathrm{K_L}$ or $\mathrm{K_S}$ only terms proportional to r^2 or ηr remain. Conventional regeneration experiments dealing with $\mathrm{K_L}$ beams are therefore only sensitive to the second order of r [30].We have included second order effects in our calculations so that Eq. 3.35 applies to all possible experimental situations.

3.3 Incoherent regeneration

For scattering angles greater than 10^{-7} the neutral kaon amplitudes scattered by two separate nuclei are no longer in phase and therefore add up *incoherently* (cf. the discussion at the beginning of the preceding Subsection). The $\mathrm{K_S}$ amplitude regenerated in this way does not interfere with the inherent

K_S component of the incoming beam which is why this type of regeneration plays a minor role in the CPLEAR experiment. Nevertheless we must assess its influence on our measurement.

Our phenomenological approach is based on the analysis of Good [28] but presented in the modernized notation introduced in the previous Subsection. We then examine the effect of incoherent regeneration for the specific case of the measurement of $\pi^+\pi^-$ decay rates with CPLEAR.

3.3.1 *Phenomenology*

We describe the scattering of K^0 and $\overline{K}^0$ at an angle θ by the amplitudes $f(\theta)$ and $\overline{f}(\theta)$. Since in our phase convention[8] $K_S = \frac{1}{\sqrt{2}}(K^0 - \overline{K}^0)$ the differential cross section for the scattering of the state ψ (see Eq. 3.9) into the final state K_S is given by

$$\frac{d\sigma_{\psi \to K_S}}{d\Omega}(\theta) = \left| \frac{\alpha f(\theta) - \overline{\alpha}\overline{f}(\theta)}{\sqrt{2}} \right|^2 = \left| \frac{\alpha_L[f(\theta) - \overline{f}(\theta)] + \alpha_S[f(\theta) + \overline{f}(\theta)]}{2} \right|^2 \tag{3.36a}$$

where the α coefficients are evaluated at the scattering point according to (3.31). By analogy we also get

$$\frac{d\sigma_{\psi \to K_L}}{d\Omega}(\theta) = \left| \frac{\alpha f(\theta) + \overline{\alpha}\overline{f}(\theta)}{\sqrt{2}} \right|^2 = \left| \frac{\alpha_L[f(\theta) + \overline{f}(\theta)] + \alpha_S[f(\theta) - \overline{f}(\theta)]}{2} \right|^2 . \tag{3.36b}$$

To obtain the number of K_S (K_L) scattered at θ into a solid angle $d\Omega$ and originating from a regenerator slice of thickness dx at position x we multiply by $N dx$, the area density of scattering centres in dx:

$$dn_S(x,\theta) = \frac{1}{4} \left| \alpha_L(x)[f(\theta) - \overline{f}(\theta)] + \alpha_S(x)[f(\theta) + \overline{f}(\theta)] \right|^2 N dx d\Omega$$
$$dn_L(x,\theta) = \frac{1}{4} \left| \alpha_L(x)[f(\theta) + \overline{f}(\theta)] + \alpha_S(x)[f(\theta) - \overline{f}(\theta)] \right|^2 N dx d\Omega$$

We integrate over the solid angle $d\Omega = 2\pi\theta d\theta$ and replace distance by proper time according to $dx = \gamma v d\tau$ to arrive at

$$dn_S(\tau_{sc}) = \frac{\pi}{2} \gamma v N d\tau_{sc} \int d\theta\, \theta\, \varepsilon(\theta) \left| \alpha_L(\tau_{sc})[f(\theta) - \overline{f}(\theta)] + \alpha_S(\tau_{sc})[f(\theta) + \overline{f}(\theta)] \right|^2 ,$$
$$dn_L(\tau_{sc}) = \frac{\pi}{2} \gamma v N d\tau_{sc} \int d\theta\, \theta\, \varepsilon(\theta) \left| \alpha_L(\tau_{sc})[f(\theta) + \overline{f}(\theta)] + \alpha_S(\tau_{sc})[f(\theta) - \overline{f}(\theta)] \right|^2 ,$$

where τ_{sc} stands for the time of the scattering and $\varepsilon(\theta)$ is the overall detection efficiency as a function of the deflection angle θ. Any K_S or K_L emerging from such a scattering process is subject to further incoherent or coherent scattering as well as absorption or decay. For our purpose, thanks to the modest size and density of the regenerator installed in the CPLEAR detector (2.5 cm $\times$ 1.85 g/cm^3 carbon, see Section 5), we may neglect multiple scattering (i.e. further incoherent regeneration) [35]. This applies all the more to scattering in the detector walls, of course. The remaining three processes, coherent scattering, absorption and decay, are well described in the framework of coherent regeneration.

3.3.2 *Contribution to the measured two pion decay rate*

Let us assume, as in Paragraph 3.2.4, that the kaons perpendicularly penetrate a regenerator of finite thickness L (Fig. 1). For a given momentum $p = \gamma v m$ they will enter the medium at proper time τ_1 and leave it again at τ_2. We are interested in the number of additional two pion decays at some proper time $\tau \geq \tau_2$ caused by incoherent regeneration. This contribution to two pion decays is given by the number of particles times their two-pionic decay rate. For K_S incoherently produced at τ_{sc} for instance, the (normalized) contribution at τ is

$$dn_S(\tau_{sc}) R_{2\pi}^{(S)}(\tau; \tau_{sc}, \tau_2)$$

8. CP violation may be ignored at this point as it cancels anyway in the right hand side of Eq. 3.36.

where

$$R_{2\pi}^{(\mathrm{S})}(\tau;\tau_{\mathrm{sc}},\tau_2) = \left|\alpha_{\mathrm{S}}^{(\mathrm{S})}(\tau) + \eta\alpha_{\mathrm{L}}^{(\mathrm{S})}(\tau)\right|^2$$

and the superscripts of the amplitudes α indicate that they are evaluated according to (3.31) for a $\mathrm{K_S}$ at the scattering time τ_{sc}: Starting from

$$\alpha_{\mathrm{S}}^{(\mathrm{S})}(\tau_{\mathrm{sc}}) = 1 \quad \text{and} \quad \alpha_{\mathrm{L}}^{(\mathrm{S})}(\tau_{\mathrm{sc}}) = 0$$

we get, if we neglect the additional path length inside the medium resulting from the scattering angle,

$$\begin{aligned}
\alpha_{\mathrm{S}}^{(\mathrm{S})}(\tau_2) &= \mathrm{e}^{-\mathrm{i}\Sigma\cdot(\tau_2-\tau_{\mathrm{sc}})}\left[\cos(\Omega(\tau_2-\tau_{\mathrm{sc}})) + \frac{\mathrm{i}\Delta\lambda}{2\Omega}\sin(\Omega(\tau_2-\tau_{\mathrm{sc}}))\right],\\
\alpha_{\mathrm{L}}^{(\mathrm{S})}(\tau_2) &= -\mathrm{e}^{-\mathrm{i}\Sigma\cdot(\tau_2-\tau_{\mathrm{sc}})}\left[\frac{\mathrm{i}\Delta\varkappa}{2\Omega}\sin(\Omega(\tau_2-\tau_{\mathrm{sc}}))\right]
\end{aligned}$$

and simply

$$\begin{aligned}
\alpha_{\mathrm{S}}^{(\mathrm{S})}(\tau) &= \mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{S}}(\tau-\tau_2)}\alpha_{\mathrm{S}}^{(\mathrm{S})}(\tau_2),\\
\alpha_{\mathrm{L}}^{(\mathrm{S})}(\tau) &= \mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{L}}(\tau-\tau_2)}\alpha_{\mathrm{L}}^{(\mathrm{S})}(\tau_2)
\end{aligned}$$

for the trajectory after the regenerator. To avoid any confusion in the final result we reserve the Greek letter α for the amplitudes of the unscattered (i.e. only coherently regenerated) beam and rename the amplitudes of the scattered particles in the following manner:

$$\begin{aligned}
a_{\mathrm{LL}} &:= \alpha_{\mathrm{L}}^{(\mathrm{L})} & a_{\mathrm{LS}} &:= \alpha_{\mathrm{L}}^{(\mathrm{S})}\\
a_{\mathrm{SL}} &:= \alpha_{\mathrm{S}}^{(\mathrm{L})} & a_{\mathrm{SS}} &:= \alpha_{\mathrm{S}}^{(\mathrm{S})}
\end{aligned}$$

The total number of two pion decays from incoherently regenerated kaons is given by the sum of the contributions from $\mathrm{K_S}$ and $\mathrm{K_L}$ integrated over the thickness of the regenerator. The final result is

$$R_{2\pi}^{\mathrm{inc}}(\tau) = \frac{\pi}{2}\gamma v N\int_{\tau_1}^{\tau_2}\mathrm{d}\tau_{\mathrm{sc}}\left(R_{2\pi}^{(\mathrm{L})}(\tau;\tau_{\mathrm{sc}},\tau_2)I_{\mathrm{L}}(\tau_1,\tau_{\mathrm{sc}}) + R_{2\pi}^{(\mathrm{S})}(\tau;\tau_{\mathrm{sc}},\tau_2)I_{\mathrm{S}}(\tau_1,\tau_{\mathrm{sc}})\right) \tag{3.37}$$

with

$$\begin{aligned}
R_{2\pi}^{(\mathrm{L})}(\tau;\tau_{\mathrm{sc}},\tau_2) &= |a_{\mathrm{SL}}(\tau;\tau_{\mathrm{sc}},\tau_2) + \eta a_{\mathrm{LL}}(\tau;\tau_{\mathrm{sc}},\tau_2)|^2,\\
R_{2\pi}^{(\mathrm{S})}(\tau;\tau_{\mathrm{sc}},\tau_2) &= |a_{\mathrm{SS}}(\tau;\tau_{\mathrm{sc}},\tau_2) + \eta a_{\mathrm{LS}}(\tau;\tau_{\mathrm{sc}},\tau_2)|^2,
\end{aligned}$$

$$I_{\mathrm{L,S}}(\tau_1,\tau_{\mathrm{sc}}) = \int\mathrm{d}\theta\,\theta\,\varepsilon(\theta)\left|\alpha_{\mathrm{L,S}}(\tau_{\mathrm{sc}})[f(\theta)+\overline{f}(\theta)] + \alpha_{\mathrm{S,L}}(\tau_{\mathrm{sc}})[f(\theta)-\overline{f}(\theta)]\right|^2 \tag{3.38}$$

and

$$\begin{aligned}
a_{\mathrm{LL}}(\tau;\tau_{\mathrm{sc}},\tau_2) &= \mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{L}}(\tau-\tau_2)}\mathrm{e}^{-\mathrm{i}\Sigma\cdot(\tau_2-\tau_{\mathrm{sc}})}\left[\cos(\Omega(\tau_2-\tau_{\mathrm{sc}})) - \frac{\mathrm{i}\Delta\lambda}{2\Omega}\sin(\Omega(\tau_2-\tau_{\mathrm{sc}}))\right],\\
a_{\mathrm{LS}}(\tau;\tau_{\mathrm{sc}},\tau_2) &= -\mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{L}}(\tau-\tau_2)}\mathrm{e}^{-\mathrm{i}\Sigma\cdot(\tau_2-\tau_{\mathrm{sc}})}\frac{\mathrm{i}\Delta\varkappa}{2\Omega}\sin(\Omega(\tau_2-\tau_{\mathrm{sc}})),\\
a_{\mathrm{SL}}(\tau;\tau_{\mathrm{sc}},\tau_2) &= -\mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{S}}(\tau-\tau_2)}\mathrm{e}^{-\mathrm{i}\Sigma\cdot(\tau_2-\tau_{\mathrm{sc}})}\frac{\mathrm{i}\Delta\varkappa}{2\Omega}\sin(\Omega(\tau_2-\tau_{\mathrm{sc}})),\\
a_{\mathrm{SS}}(\tau;\tau_{\mathrm{sc}},\tau_2) &= \mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{S}}(\tau-\tau_2)}\mathrm{e}^{-\mathrm{i}\Sigma\cdot(\tau_2-\tau_{\mathrm{sc}})}\left[\cos(\Omega(\tau_2-\tau_{\mathrm{sc}})) + \frac{\mathrm{i}\Delta\lambda}{2\Omega}\sin\left(\Omega(\tau_2-\tau_{\mathrm{sc}})\right)\right].
\end{aligned}$$

$\alpha_L(\tau_{sc})$ and $\alpha_S(\tau_{sc})$ are the coherently regenerated (unscattered) amplitudes at the time of the scattering τ_{sc}, recursively calculated with (3.31) as

$$\alpha_L(\tau_{sc}) = e^{-i\Sigma\cdot(\tau_{sc}-\tau_1)}\left[\alpha_L(\tau_1)\cos(\Omega(\tau_{sc}-\tau_1)) - \frac{i}{2\Omega}(\Delta\lambda\alpha_L(\tau_1) + \Delta\varkappa\alpha_S(\tau_1))\sin(\Omega(\tau_{sc}-\tau_1))\right],$$
$$\alpha_S(\tau_{sc}) = e^{-i\Sigma\cdot(\tau_{sc}-\tau_1)}\left[\alpha_S(\tau_1)\cos(\Omega(\tau_{sc}-\tau_1)) + \frac{i}{2\Omega}(\Delta\lambda\alpha_S(\tau_1) - \Delta\varkappa\alpha_L(\tau_1))\sin(\Omega(\tau_{sc}-\tau_1))\right];$$

$$\alpha_L(\tau_1) = e^{-i\lambda_L\tau_1}\alpha_L(0),$$
$$\alpha_S(\tau_1) = e^{-i\lambda_S\tau_1}\alpha_S(0),$$

from the initial amplitudes $\alpha_L(0)$ and $\alpha_S(0)$ as given by (3.8) for initial K^0 and $\overline{K}^0$. $R_{2\pi}^{inc}(\tau)$ is to be added to the decay rate after coherent regeneration, $R_{2\pi}^{coh}(\tau)$, to obtain the observed decay rate:

$$R_{2\pi}^{obs}(\tau) = R_{2\pi}^{coh}(\tau) + R_{2\pi}^{inc}(\tau) \tag{3.39}$$

For all practical purposes the contribution of K_L to the 2π decay rate may safely be neglected and (3.37) reduces to

$$R_{2\pi}^{inc}(\tau) = \frac{\pi}{2}\gamma v N \int_{\tau_1}^{\tau_2} d\tau_{sc}|\alpha_{SS}(\tau;\tau_{sc},\tau_2)|^2 I_S(\tau_1,\tau_{sc}). \tag{3.40}$$

3.3.3 *Numerical computation for $\pi^+\pi^-$ decays at CPLEAR*

In order to carry out the integration (3.40) we must know the angular variation of the scattering amplitudes $f(\theta)$, $\overline{f}(\theta)$ and the experimental detection efficiency $\varepsilon(\theta)$.

$f(\theta)$ and $\overline{f}(\theta)$ can be well reproduced by an optical model calculation which results in an integration over a weighted Bessel function. The weighing function to be applied is not the same for K^0 and $\overline{K}^0$ reflecting the different interaction length of the two states inside the nucleus and therefore $f(\theta)$ and $\overline{f}(\theta)$ have somewhat different slopes [36]. For our estimate, however, we do not take into account this rather small difference [37] and assume furthermore that only the magnitudes of the amplitudes are affected by the angular variation:

$$\frac{\overline{f}(\theta)}{\overline{f}(0)} \approx \frac{f(\theta)}{f(0)} \approx \frac{|f(\theta)|}{|f(0)|} =: F(\theta)$$

This last assumption is not valid in the vicinity of resonances, but the deviations will not significantly alter our conclusions. The θ integrations (3.38) then simplify to

$$I_{L,S}(\tau_1,\tau_{sc}) = \left|\alpha_{L,S}(\tau_{sc})[f(0)+\overline{f}(0)] + \alpha_{S,L}(\tau_{sc})[f(0)-\overline{f}(0)]\right|^2 I_\theta$$

with

$$I_\theta = \int d\theta\,\theta\,\varepsilon(\theta)|F(\theta)|^2. \tag{3.41}$$

For $f(0)$ and $\overline{f}(0)$ we take the theoretical results of Baldini and Michetti [38]. Their values are in agreement with the work of Eberhard and Uchiyama [39] who give the difference $f(0) - \overline{f}(0)$ only.

We then describe $|F(\theta)|^2$ by a Gauss function [30, 36],

$$|F(\theta)|^2 \approx \exp\left[-\frac{1}{2}\left(\frac{\theta}{\sigma_f}\right)^2\right]. \tag{3.42}$$

We expect the half-width σ_f to be roughly inversely proportional to the momentum. Experimental data exist for $K^\pm$ scattering on carbon nuclei at 800 MeV/*c* [37] and indicate $\sigma_f \approx 6.8°$. For lower momenta, we use calculations of elastic K_L scattering [40] to estimate σ_f. At 500 MeV/*c* we find a value of $\approx 9.8°$.

For $\pi^+\pi^-$ decays, our results depend only weakly on the exact angular behaviour of the scattering amplitudes because the integral I_θ is dominated by the detection efficiency.

The angular detection efficiency of neutral kaons decaying to $\pi^+\pi^-$ for the CPLEAR experiment, i.e. the probability for a scattered $K^0(\overline{K}^0)$ to survive all selection criteria including the constrained fit for $\pi^+\pi^-$ decays (9C-fit, see Section 6.2), was first studied by Ch. Yeche [41], and more recently and more thoroughly by Ph. Bloch [42]. The computed acceptance curves are fairly well approximated by the sum of two Gaussians,

$$\varepsilon(\theta) \approx \alpha \exp\left[-\frac{1}{2}\left(\frac{\theta}{\sigma_1}\right)^2\right] + (1-\alpha)\exp\left[-\frac{1}{2}\left(\frac{\theta}{\sigma_2}\right)^2\right], \tag{3.43}$$

where the values of the parameters α, σ_1 and σ_2 depend on the radius at which the scattering occurs. For a scatterer at a distance of about 10 cm for instance we find $\alpha \approx 0.8$, $\sigma_1 \approx 1.7°$ and $\sigma_2 \approx 3.5°$; a smaller distance leads to smaller widths.

Inserting the numbers for various energies and scattering radii into (3.42) and (3.43) yields values for the θ integration (3.41) between 1 and 2.5×10^{-3} [43]. For the carbon regenerator used in our experiment we find an average of

$$I_\theta \approx 1.35 \times 10^{-3}. \tag{3.44}$$

Having made these simplifications it is a straightforward (albeit time consuming) task to compute the number of additional $\pi^+\pi^-$ decays caused by incoherent regeneration for any geometry as long as we can compute the time the particle spends inside the regenerator.