Phenomenology of the K^0–$\overline{K}^0$ System

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Abstract

The CPLEAR experiment measures $\mathcal{T}$, $\mathcal{CP}$ and $\mathcal{CPT}$ violation parameters in the neutral kaon system through time dependent decay rate asymmetries of $\mathcal{T}$, $\mathcal{CP}$ and $\mathcal{CPT}$ conjugated processes. The $K^0-\overline{K}^0$ phenomenology used to describe the CPLEAR measurements is presented without any phase convention and without assuming from the beginning small violations of $\mathcal{T}$, $\mathcal{CP}$ or $\mathcal{CPT}$ symmetry.

1 Foreword

This note will hopefully improve in future with all the errors found by the readers corrected and the parts rewritten which are still not explained clear enough. I have not choosen a specific phase convention, since without a phaseconvention, it is much easier to decide which parameters can be measured and which parameters depend on a specific phaseconvention. It was also not necessary to start with assuming small $\mathcal{CP}$ violation. The basic formulae do not become more complicated. All calculations are based on the strangeness eigenstates K^0 and $\overline{K}^0$.

2 Phenomenology of the neutral Kaon System

In the absence of any strangeness violating interaction, the stationary states, $|K^0\rangle$ and $|\overline{K}^0\rangle$, of a K^0-meson and $\overline{K}^0$-meson respectively, are mass eigenstates of the strong and electromagnetic interactions:

$$(\mathcal{H}_{st} + \mathcal{H}_{em})|K^0\rangle = m_o|K^0\rangle \qquad (\mathcal{H}_{st} + \mathcal{H}_{em})|\overline{K}^0\rangle = m_o|\overline{K}^0\rangle \tag{1}$$

Since the strangeness violating interaction $\mathcal{H}_{weak}$ is much weaker than the strong and electromagnetic interaction we can apply perturbation theory (Wigner-Weisskopf approach) and obtain for the time evolution of the neutral kaon wave function the following differential equation:

$$i\frac{\partial}{\partial\tau}\begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix} = \Lambda \begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix} = (M - \frac{i}{2}\Gamma)\begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix} \tag{2}$$

where the 2×2 matrices M and Γ are called mass and decay matrix respectively and given by:

$$M_{ij} = m_o\delta_{ij} + \langle i|\mathcal{H}_{weak}|j\rangle + \sum_f \mathcal{P}\left(\frac{\langle i|\mathcal{H}_{weak}|f\rangle\langle f|\mathcal{H}_{weak}|j\rangle}{m_o - E_f}\right) \tag{3}$$

$$\Gamma_{ij} = 2\pi\sum_f \langle i|\mathcal{H}_{weak}|f\rangle\langle f|\mathcal{H}_{weak}|j\rangle\delta(m_o - E_f) \tag{4}$$

with the index $i = 1(2)$ corresponding to $\mathrm{K}^0(\overline{\mathrm{K}}^0)$. The matrix elements of Λ can in principle be calculated within the Standard Model but in practice the uncertainties due to non-perturbative effects are too large to obtain any interesting predictions. Therefore we use the following parametrisation of Λ with 8 real and positive parameters:

$$\Lambda = \begin{pmatrix} M_{11} & M_{12}\mathrm{e}^{\mathrm{i}\varphi_{\mathrm{M}}} \\ M_{12}\mathrm{e}^{-\mathrm{i}\varphi_{\mathrm{M}}} & M_{22} \end{pmatrix} - \frac{i}{2}\begin{pmatrix} \Gamma_{11} & \Gamma_{12}\mathrm{e}^{\mathrm{i}\varphi_\Gamma} \\ \Gamma_{12}\mathrm{e}^{-\mathrm{i}\varphi_\Gamma} & \Gamma_{22} \end{pmatrix} \tag{5}$$

where M_{11}, M_{22}, Γ_{11}, Γ_{22} are equal to the masses and lifetimes of K^0 and $\overline{\mathrm{K}}^0$ respectively.

2.1 Discrete Symmetries

The two discrete transformations $\mathcal{CP}$ and $\mathcal{T}$ in the neutral kaon system are defined by:

$$\begin{array}{rclcrcl} \mathcal{CP}|\mathrm{K}^0\rangle &=& \mathrm{e}^{\mathrm{i}\phi_{\mathrm{CP}}}|\overline{\mathrm{K}}^0\rangle & \quad & \mathcal{CP}|\overline{\mathrm{K}}^0\rangle &=& \mathrm{e}^{-\mathrm{i}\phi_{\mathrm{CP}}}|\mathrm{K}^0\rangle \\ \mathcal{T}|\mathrm{K}^0\rangle &=& \mathrm{e}^{\mathrm{i}\phi_{\mathrm{T}}}|\mathrm{K}^0\rangle & & \mathcal{T}|\overline{\mathrm{K}}^0\rangle &=& \mathrm{e}^{\mathrm{i}\overline{\phi}_{\mathrm{T}}}|\overline{\mathrm{K}}^0\rangle \end{array} \tag{6}$$

where we remind that $\mathcal{T}$ is an antiunitary tansformation. According to the $\mathcal{CPT}$ theorem, the arbitrary phases ϕ_T, $\phi_{\overline{T}}$, ϕ_{CP}, can be choosen in such a way that the interaction is invariant under $\mathcal{CPT}$ transformation. We obtain the following conditions:

$$\left.\Lambda_{11} = \Lambda_{22} \text{ and } 2\phi_{\mathrm{CP}} = \overline{\phi}_{\mathrm{T}} - \phi_{\mathrm{T}}\right\} \begin{array}{l} M_{11} = M_{22} \\ \Gamma_{11} = \Gamma_{22} \end{array} \tag{7}$$

which are equivalent to equal masses and lifetimes for K^0 and $\overline{\mathrm{K}}^0$. Assuming $\mathcal{T}$ invariance of the interaction we obtain the following conditions:

$$\left.|\Lambda_{12}| = |\Lambda_{21}| \text{ and } \phi_{\mathrm{C}} = -\varphi_\Gamma + \mathrm{n}\cdot\pi\right\} \sin(\varphi_{\mathrm{M}} - \varphi_\Gamma) = 0 \tag{8}$$

which are equivalent to equal transition rates $\mathrm{K}^0 \to \overline{\mathrm{K}}^0$ and $\overline{\mathrm{K}}^0 \to \mathrm{K}^0$. $\mathcal{CP}$ invariance in the neutral kaon system requires $\mathcal{T}$ and $\mathcal{CPT}$ invariance both to be fulfilled.

2.2 Time Evolution

The time evolution of an arbitrary neutral kaon state after solving eq. 2 is given by:

$$a(\tau)|\mathrm{K}^0\rangle + b(\tau)|\overline{\mathrm{K}}^0\rangle = \begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix} = T(\tau)\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} \tag{9}$$

$$\text{with}\quad \mathrm{T}(\tau) = \begin{pmatrix} \frac{\Lambda_{22}-\Lambda_{11}}{\Delta\lambda} f_-(\tau) + f_+(\tau) & -2\frac{\Lambda_{12}}{\Delta\lambda} f_-(\tau) \\ -2\frac{\Lambda_{21}}{\Delta\lambda} f_-(\tau) & -\frac{\Lambda_{22}-\Lambda_{11}}{\Delta\lambda} f_-(\tau) + f_+(\tau) \end{pmatrix} \tag{10}$$

$$\text{and}\quad \mathrm{T}(\tau_1)\cdot\mathrm{T}(\tau_2) = \mathrm{T}(\tau_2)\cdot\mathrm{T}(\tau_1) = \mathrm{T}(\tau_1+\tau_2)$$

with

$$\begin{aligned} f_+(\tau) &= \frac{e^{-i\lambda_S\tau} + e^{-i\lambda_L\tau}}{2} \qquad f_-(\tau) = \frac{e^{-i\lambda_S\tau} - e^{-i\lambda_L\tau}}{2} \\ \lambda_{L,S} &= \frac{\Lambda_{11}+\Lambda_{22}}{2} \pm \sqrt{\frac{(\Lambda_{22}-\Lambda_{11})^2}{4} + \Lambda_{12}\Lambda_{21}} \\ &= m_{L,S} - \frac{i}{2}\Gamma_{L,S} \\ \Delta\lambda &= \lambda_L - \lambda_S = \sqrt{(\Lambda_{22}-\Lambda_{11})^2 + 4\Lambda_{12}\Lambda_{21}} \end{aligned}$$

where λ_S and λ_L are the eigenvalues of the matrix Λ. The eigenvectors which follow an exponential decay law are given by:

$$\begin{aligned} |\mathrm{K_S}\rangle &= \frac{e^{i\varphi_S}}{\sqrt{1+|r_S|^2}}\left(r_S|\mathrm{K}^0\rangle + |\overline{\mathrm{K}}^0\rangle\right) \\ |\mathrm{K_L}\rangle &= \frac{e^{i\varphi_L}}{\sqrt{1+|r_L|^2}}\left(r_L|\mathrm{K}^0\rangle + |\overline{\mathrm{K}}^0\rangle\right) \end{aligned} \tag{11}$$

with arbitrary phases φ_S, φ_L and

$$r_S = \frac{2\Lambda_{12}}{\Lambda_{22}-\Lambda_{11}-\Delta\lambda}, \qquad r_L = \frac{2\Lambda_{12}}{\Lambda_{22}-\Lambda_{11}+\Delta\lambda}$$

In the special case of initial pure $|\mathrm{K}^0\rangle$ or $|\overline{\mathrm{K}}^0\rangle$ at time $\tau = 0$ we obtain:

$$\begin{aligned} |\mathrm{K}^0(\tau)\rangle &= \left[\frac{\Lambda_{22}-\Lambda_{11}}{\Delta\lambda} f_-(\tau) + f_+(\tau)\right]|\mathrm{K}^0\rangle - 2\frac{\Lambda_{21}}{\Delta\lambda} f_-(\tau)|\overline{\mathrm{K}}^0\rangle \\ &= e^{-i\lambda_S\tau}\left(\left(\frac{\Lambda_{22}-\Lambda_{11}}{2\Delta\lambda} + \frac{1}{2}\right)|\mathrm{K}^0\rangle - \frac{\Lambda_{21}}{\Delta\lambda}|\overline{\mathrm{K}}^0\rangle\right) \\ &\quad + e^{-i\lambda_L\tau}\left(\left(-\frac{\Lambda_{22}-\Lambda_{11}}{2\Delta\lambda} + \frac{1}{2}\right)|\mathrm{K}^0\rangle + \frac{\Lambda_{21}}{\Delta\lambda}|\overline{\mathrm{K}}^0\rangle\right) \\ |\overline{\mathrm{K}}^0(\tau)\rangle &= -2\frac{\Lambda_{12}}{\Delta\lambda} f_-(\tau)|\mathrm{K}^0\rangle + \left[-\frac{\Lambda_{22}-\Lambda_{11}}{\Delta\lambda} f_-(\tau) + f_+(\tau)\right]|\overline{\mathrm{K}}^0\rangle \end{aligned} \tag{12}$$

$$
\begin{aligned}
&= e^{-i\lambda_S\tau}\left((-\frac{\Lambda_{22}-\Lambda_{11}}{2\Delta\lambda}+\frac{1}{2})|\overline{\mathrm{K}}^0\rangle-\frac{\Lambda_{12}}{\Delta\lambda}|\mathrm{K}^0\rangle\right)\\
&\quad+e^{-i\lambda_L\tau}\left((\frac{\Lambda_{22}-\Lambda_{11}}{2\Delta\lambda}+\frac{1}{2})|\overline{\mathrm{K}}^0\rangle+\frac{\Lambda_{12}}{\Delta\lambda}|\mathrm{K}^0\rangle\right)
\end{aligned}
\tag{13}
$$

In the limit of small $\mathcal{CPT}$ violation, i.e. $|\frac{\Lambda_{22}-\Lambda_{11}}{\Delta\lambda}| \ll 1$ and small $\mathcal{T}$ violation, i.e. $|\frac{|\Lambda_{12}|^2-|\Lambda_{21}|^2}{2\Delta\Gamma\Delta\mathrm{m}}| \ll 1$, we obtain:

$$
\begin{aligned}
|\mathrm{K}^0(\tau)\rangle &= [-2\delta f_-(\tau)+f_+(\tau)]\,|\mathrm{K}^0\rangle+(1-2\varepsilon_\mathrm{T})\,\mathrm{e}^{-\mathrm{i}\varphi_\Gamma}f_-(\tau)|\overline{\mathrm{K}}^0\rangle\\
&= \frac{1}{2}\left[(1-2\delta)|\mathrm{K}^0\rangle+(1-2\varepsilon_\mathrm{T})\mathrm{e}^{-\mathrm{i}\varphi_\Gamma}|\overline{\mathrm{K}}^0\rangle\right]e^{-i\lambda_S\tau}\\
&\quad+\frac{1}{2}\left[(1+2\delta)|\mathrm{K}^0\rangle-(1-2\varepsilon_\mathrm{T})\mathrm{e}^{-\mathrm{i}\varphi_\Gamma}|\overline{\mathrm{K}}^0\rangle\right]e^{-i\lambda_L\tau}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
|\overline{\mathrm{K}}^0(\tau)\rangle &= [2\delta f_-(\tau)+f_+(\tau)]\,|\overline{\mathrm{K}}^0\rangle+(1+2\varepsilon_\mathrm{T})\,\mathrm{e}^{\mathrm{i}\varphi_\Gamma}f_-(\tau)|\mathrm{K}^0\rangle\\
&= \frac{1}{2}\left[(1+2\delta)|\overline{\mathrm{K}}^0\rangle+(1+2\varepsilon_\mathrm{T})\mathrm{e}^{\mathrm{i}\varphi_\Gamma}|\mathrm{K}^0\rangle\right]e^{-i\lambda_S\tau}\\
&\quad+\frac{1}{2}\left[(1-2\delta)|\overline{\mathrm{K}}^0\rangle-(1+2\varepsilon_\mathrm{T})\mathrm{e}^{\mathrm{i}\varphi_\Gamma}|\mathrm{K}^0\rangle\right]e^{-i\lambda_L\tau}
\end{aligned}
\tag{15}
$$

with ε_T and δ describing $\mathcal{T}$ and $\mathcal{CPT}$ violation in the mass and decay matrix:

$$
\varepsilon_\mathrm{T} \equiv \frac{\Delta\mathrm{m}\Delta\Gamma+2i\Delta\mathrm{m}^2}{4\Delta\mathrm{m}^2+\Delta\Gamma^2}\cdot\frac{|\Lambda_{12}|^2-|\Lambda_{21}|^2}{\Delta\Gamma/2\Delta\mathrm{m}} \tag{16}
$$

$$
\delta \equiv \frac{i\Delta\Gamma\Delta\mathrm{m}-2\Delta\mathrm{m}^2}{4\Delta\mathrm{m}^2+\Delta\Gamma^2}\cdot\frac{\Lambda_{22}-\Lambda_{11}}{\Delta\mathrm{m}} \tag{17}
$$

The phase of the parameter ε_T is identical to the superweak phase $\varphi_{SW} = \mathrm{atan}\dfrac{2\Delta\mathrm{m}}{\Delta\Gamma}$. The masses and decay widths of the two eigenstates are given by [1]:

$$
\begin{aligned}
m_{L,S} &= \frac{M_{11}+M_{22}}{2}\pm M_{12}, & \Gamma_{L,S} &= \frac{\Gamma_{11}+\Gamma_{22}}{2}\mp\Gamma_{12}\\
\Delta\Gamma &= \Gamma_S-\Gamma_L = 2\Gamma_{12}, & \Delta\mathrm{m} &= m_L-m_S = 2M_{12}\\
r_S &= \mathrm{e}^{\mathrm{i}\varphi_\Gamma}\left(1+2\varepsilon_\mathrm{T}-2\delta\right), & r_L &= -\mathrm{e}^{\mathrm{i}\varphi_\Gamma}\left(1+2\varepsilon_\mathrm{T}+2\delta\right)
\end{aligned}
$$

In the limit of $\mathcal{CP}$ conservation and by fixing the arbitrary phase $\phi_{CP} = -\varphi_\Gamma$, $\mathrm{K_S}$ and $\mathrm{K_L}$ become $\mathcal{CP}$ eigenstates with eigenvalues $+1$ and -1:

$$
\begin{aligned}
|\mathrm{K_S}\rangle &= \tfrac{e^{i\varphi_S}}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\varphi_\Gamma}|\mathrm{K}^0\rangle+|\overline{\mathrm{K}}^0\rangle\right) & \mathcal{CP}|\mathrm{K_S}\rangle &= +|\mathrm{K_S}\rangle\\
|\mathrm{K_L}\rangle &= \tfrac{e^{i\varphi_L}}{\sqrt{2}}\left(-\mathrm{e}^{\mathrm{i}\varphi_\Gamma}|\mathrm{K}^0\rangle+|\overline{\mathrm{K}}^0\rangle\right) & \mathcal{CP}|\mathrm{K_L}\rangle &= -|\mathrm{K_L}\rangle
\end{aligned}
$$

3 Decay Rates

The decay rates of neutral kaons are the only known example of a nonexponential decay law in nature for elementary particles.

[1] $\Delta\Gamma$ and $\Delta\mathrm{m}$ are real and positive: $\Gamma_S > \Gamma_L$ but $m_S < m_L$

3.1 Decay rate into two pions

Following the formalism presented in section 1 we obtain the time dependent decay rates for initially pure K^0 and $\overline{\mathrm{K}}^0$ in $\pi^+\pi^-$, $\pi^0\pi^0$:

$$R(\mathrm{K}^0 \to \pi\pi)(\tau) = B_{\pi\pi}\left[\mathrm{e}^{-\Gamma_\mathrm{S}\tau} + |\eta_{\pi\pi}|^2\mathrm{e}^{-\Gamma_\mathrm{L}\tau} + 2|\eta_{\pi\pi}|\mathrm{e}^{-\overline{\Gamma}\tau}\cos(\Delta\mathrm{m}\tau - \varphi_{\pi\pi})\right] \quad (18)$$
$$R(\overline{\mathrm{K}}^0 \to \pi\pi)(\tau) = \overline{B}_{\pi\pi}\left[\mathrm{e}^{-\Gamma_\mathrm{S}\tau} + |\overline{\eta}_{\pi\pi}|^2\mathrm{e}^{-\Gamma_\mathrm{L}\tau} - 2|\overline{\eta}_{\pi\pi}|\mathrm{e}^{-\overline{\Gamma}\tau}\cos(\Delta\mathrm{m}\tau - \overline{\varphi}_{\pi\pi})\right] \quad (19)$$

with $\overline{\Gamma} = \dfrac{\Gamma_S + \Gamma_L}{2}$. The parameter $\eta_{\pi\pi}$ vanishes in case of $\mathcal{CP}$ conservation, i.e. $\langle\pi\pi|T|\mathrm{K_L}\rangle \equiv 0$:

$$\begin{array}{rclrcl} \eta_{\pi\pi} & = & -\dfrac{\langle\pi\pi|T|\mathrm{K_L}\rangle}{\langle\pi\pi|T|\mathrm{K_S}\rangle}\dfrac{\sqrt{1+|r_L|^2}}{\sqrt{1+|r_S|^2}}e^{i(\varphi_S-\varphi_L)}, & \overline{\eta}_{\pi\pi} & = & -\dfrac{r_S}{r_L}\eta_{\pi\pi} \\ B_{\pi\pi} & = & |\langle\pi\pi|T|\mathrm{K_S}\rangle|^2\frac{1+|r_S|^2}{|r_S-r_L|^2}, & \overline{B}_{\pi\pi} & = & |r_L|^2 B_{\pi\pi} \end{array} \quad (20)$$

In the limit of small $\mathcal{T}$ and even smaller $\mathcal{CPT}$ violation we can write:

$$\eta_{\pi\pi} \approx \overline{\eta}_{\pi\pi} \qquad B_{\pi\pi} \approx |\langle\pi\pi|T|\mathrm{K_S}\rangle|^2\,(1 - 2\Re e\,(\varepsilon_\mathrm{T} + \delta)) \qquad \overline{B}_{\pi\pi} \approx |\langle\pi\pi|T|\mathrm{K_S}\rangle|^2\,(1 + 2\Re e\,(\varepsilon_\mathrm{T} + \delta)) \quad (21)$$

The $\mathcal{CP}$ violating parameter $\eta_{\pi\pi}$ is best measured through the interference term by constructing the following time dependent decay rate asymmetry:

$$\begin{aligned} A_{2\pi} &= \frac{R(\overline{\mathrm{K}}^0 \to \pi\pi)(\tau) - \alpha \times R(\mathrm{K}^0 \to \pi\pi)(\tau)}{R(\overline{\mathrm{K}}^0 \to \pi\pi)(\tau) + \alpha \times R(\mathrm{K}^0 \to \pi\pi)(\tau)} \\ &= -2|\eta_{\pi\pi}|\cos(\Delta\mathrm{m}\tau - \varphi_{\pi\pi})\frac{e^{\frac{1}{2}(\Gamma_S-\Gamma_L)\tau}}{1+|\eta_{\pi\pi}|^2 e^{(\Gamma_S-\Gamma_L)\tau}} \end{aligned} \quad (22)$$

with $\alpha = 1 + 4\Re e\,(\varepsilon_\mathrm{T} + \delta)$.

3.2 Decay rate into three pions

Neutral Kaons can decay into $\pi^+\pi^-\pi^0$ and $3\pi^0$ final states. For the $\pi^+\pi^-\pi^0$ decay mode, the $\pi^+\pi^-$ pair can have total isospin $I' = 0, 1$ and 2. The total wave function has to be symmetric under exchange of π^+ and π^-. If the orbital angular momentum of the $\pi^+\pi^-$ state is odd then the isospin must be $I' = 1$ and $I' = 0, 2$ for even angular momentum. The total isospin must be $I = 3, 2, 1$ or 0.

$$\begin{array}{llll} \pi^+\pi^-\pi^0: & I = 1,3 & \mathcal{CP} = -1 & l = l' = \text{even} \\ & I = 0,2 & \mathcal{CP} = +1 & l = l' = \text{odd} \\ \pi^0\pi^0\pi^0: & I = 1,3 & \mathcal{CP} = -1 & l = l' = \text{even} \end{array}$$

The decay rates of the $\mathrm{K_L}$ and $\mathrm{K_S}$ state into the 3π final state can be expressed as:

$$\begin{array}{rcl} \langle 3\pi(X,Y)|T|\mathrm{K_L}\rangle & = & A_L = \alpha + \beta Y + \xi_Y Y^2 + \xi_X X^2 \\ \langle 3\pi(X,Y)|T|\mathrm{K_S}\rangle & = & A_S = \alpha' + \beta' Y + \xi'_Y Y^2 + \xi'_X X^2 + \gamma X + \xi'_{XY} XY \end{array} \quad (23)$$

with the usual Dalitz parameters:

$$X = 2\frac{m_K}{m^2_{\pi^+}}(E^*_{\pi^+} - E^*_{\pi^-}) \qquad Y = 2\frac{m^2_K + m^2_{\pi^0} - m^2_{\pi^+}}{3m^2_{\pi^+}} - 2\frac{m_K}{m^2_{\pi^+}}E^*_{\pi^0}$$

The energies E^* are defined in the rest frame of the neutral kaon. The $\mathcal{CP}$ violating contribution to the $\mathrm{K_L}$ decay is suppressed due to the centrifugal barrier and neglected in A_L. The $\mathcal{CP}$ conserving contributions to the $\mathrm{K_S}$ decay proportional to X vanish in the case of the $3\pi^0$ final state because X is antisymmetric with respect to the exchange of π_1 and π_2. The total decay rates of K^0 and $\overline{\mathrm{K}}^0$ are given by integrating over the whole phase space:

$$\int\int |\langle 3\pi|T|\mathrm{K}^0(\tau)\rangle|^2 dX\,dY$$
$$= B_{3\pi}\left[\mathrm{e}^{-\Gamma_\mathrm{L}\tau} + |\kappa_{3\pi}|^2\mathrm{e}^{-\Gamma_\mathrm{S}\tau} + 2|\eta_{3\pi}|\mathrm{e}^{-\overline{\Gamma}\tau}\cos(\Delta\mathrm{m}\tau + \varphi_{3\pi})\right] \qquad (24)$$
$$\int\int |\langle 3\pi|T|\overline{\mathrm{K}}^0(\tau)\rangle|^2 dX\,dY$$
$$= \overline{B}_{3\pi}\left[\mathrm{e}^{-\Gamma_\mathrm{L}\tau} + |\overline{\kappa}_{3\pi}|^2\mathrm{e}^{-\Gamma_\mathrm{S}\tau} - 2|\overline{\eta}_{3\pi}|\mathrm{e}^{-\overline{\Gamma}\tau}\cos(\Delta\mathrm{m}\tau + \overline{\varphi}_{3\pi})\right] \qquad (25)$$

$$\begin{aligned}
B_{3\pi} &= \tfrac{1+|r_L|^2}{|r_S-r_L|^2}\int\int|A_L|^2 dX\,dY & \overline{B}_{3\pi} &= |r_S|^2 B_{3\pi}\\
|\kappa_{3\pi}|^2 &= \tfrac{1+|r_S|^2}{1+|r_L|^2}\tfrac{\int\int|A_S|^2 dX\,dY}{\int\int|A_L|^2 dX\,dY} & |\overline{\kappa}_{3\pi}|^2 &= |\tfrac{r_L}{r_S}|^2\kappa_{3\pi}\\
\eta_{3\pi} &= -\tfrac{\sqrt{1+|r_S|^2}}{\sqrt{1+|r_L|^2}}e^{i(\varphi_L-\varphi_S)}\tfrac{\int\int\langle 3\pi(X,Y)|T|\mathrm{K_S}\rangle\langle 3\pi(X,Y)|T|\mathrm{K_L}\rangle^* dX\,dY}{\int\int|A_L|^2 dX\,dY} & &\\
\overline{\eta}_{3\pi} &= -\tfrac{r_L}{r_S}\eta_{3\pi} & &
\end{aligned}$$

In this case $\eta_{3\pi}$ becomes zero if there is no $\mathcal{CP}$ violation in $\mathrm{K_S}$ decays whereas $\kappa_{3\pi}$ in addition to the $\mathcal{CP}$ violating amplitude contains also the $\mathrm{K_S}$ $\mathcal{CP}$ conserving amplitude for the $\pi^+\pi^-\pi^0$ final state. In the limit of small $\mathcal{CPT}$ and $\mathcal{T}$ violation we get $\overline{\eta}_{3\pi} \approx \eta_{3\pi}$ and $\overline{\kappa}_{3\pi} \approx \kappa_{3\pi}$. The interference term can be extracted using the following time dependent decay rate asymmetry:

$$A_{3\pi}(\tau) = \frac{\overline{R}(\tau) - \alpha\times R(\tau)}{\overline{R}(\tau) + \alpha\times R(\tau)} = -2\frac{|\eta_{3\pi}|e^{-(\Gamma_S-\Gamma_L)\tau}\cos(\Delta\mathrm{m}\tau + \varphi_{3\pi})}{1 + |\kappa_{3\pi}|^2 e^{-\frac{1}{2}(\Gamma_S-\Gamma_L)\tau}}$$

where $\alpha = 1 + 4\Re e\,(\varepsilon_\mathrm{T} - \delta)$. Even with the contribution of the $\mathcal{CP}$ conserving amplitude in the $\pi^+\pi^-\pi^0$ decay we can write the asymmetry to a good approximation as:

$$A_{3\pi}(\tau) \sim -2|\eta_{3\pi}|\cos(\Delta\mathrm{m}\tau + \varphi_{3\pi})e^{-\frac{1}{2}(\Gamma_S-\Gamma_L)t} \qquad (26)$$

3.3 Semileptonic decay rates

The rates $\mathrm{K}^0 \rightarrow l^- X$ and $\overline{\mathrm{K}}^0 \rightarrow l^+ X$ are very much suppressed within the Standard Model. This is known as the $\Delta S = \Delta Q$ rule. We will discuss in Appendix A the modifications to the semileptonic decay rates in case of a new $\Delta S = \Delta Q$ violating interaction. The four measured decay rates are:

$$\begin{aligned}
R(\mathrm{K}^0{}_{\tau=0} \rightarrow l^+)(\tau) &= B_+ \Big\{ [1 - 4\Re e\,(\delta)]\, \mathrm{e}^{-\Gamma_\mathrm{S}\tau} + [1 + 4\Re e\,(\delta)]\, \mathrm{e}^{-\Gamma_\mathrm{L}\tau} \\
&\quad + [2\cos \Delta \mathrm{m}\tau + 8\Im m\,(\delta) \sin \Delta \mathrm{m}\tau]\, \mathrm{e}^{-\overline{\Gamma}\tau} \Big\} && (27) \\
R(\overline{\mathrm{K}}^0{}_{\tau=0} \rightarrow l^-)(\tau) &= B_- \Big\{ [1 + 4\Re e\,(\delta)]\, \mathrm{e}^{-\Gamma_\mathrm{S}\tau} + [1 - 4\Re e\,(\delta)]\, \mathrm{e}^{-\Gamma_\mathrm{L}\tau} \\
&\quad + [2\cos \Delta \mathrm{m}\tau - 8\Im m\,(\delta) \sin \Delta \mathrm{m}\tau]\, \mathrm{e}^{-\overline{\Gamma}\tau} \Big\} && (28) \\
R(\mathrm{K}^0{}_{\tau=0} \rightarrow l^-)(\tau) &= B_- \Big\{ [1 - 4\Re e\,(\varepsilon_\mathrm{T})]\, \mathrm{e}^{-\Gamma_\mathrm{S}\tau} + [1 - 4\Re e\,(\varepsilon_\mathrm{T})]\, \mathrm{e}^{-\Gamma_\mathrm{L}\tau} \\
&\quad - 2\,[1 - 4\Re e\,(\varepsilon_\mathrm{T})]\, \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau \Big\} && (29) \\
R(\overline{\mathrm{K}}^0{}_{\tau=0} \rightarrow l^+)(\tau) &= B_+ \Big\{ [1 + 4\Re e\,(\varepsilon_\mathrm{T})]\, \mathrm{e}^{-\Gamma_\mathrm{S}\tau} + [1 + 4\Re e\,(\varepsilon_\mathrm{T})]\, \mathrm{e}^{-\Gamma_\mathrm{L}\tau} \\
&\quad - 2\,[1 + 4\Re e\,(\varepsilon_\mathrm{T})]\, \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau \Big\} && (30)
\end{aligned}$$

3.4 Direct measurements of $\mathcal{T}$ and $\mathcal{CPT}$ violation

These measurements are possible by using semileptonic decays of neutral kaons and assuming the validity of the $\Delta S = \Delta Q$ rule. Due to a possible new interaction violating the $\Delta S = \Delta Q$ rule the above formula have to be modified according to Appendix A. If the strangeness of the neutral kaon can be tagged at the time of its production and at its decay we can measure the rate of a K^0 oscillating in a $\overline{\mathrm{K}}^0$ and its $\mathcal{T}$-conjugated process, $\overline{\mathrm{K}}^0$ oscillating in a K^0 as well as the rate that a K^0 decayed as a K^0 compared to its $\mathcal{CPT}$ conjugated process. The decay rate asymmetries constructed with these four rates are a direct measurement of the $\mathcal{T}$ and $\mathcal{CPT}$ violation in the $\mathrm{K}^0 - \overline{\mathrm{K}}^0$ mass and decay matrix:

$$\begin{aligned}
A_T(\tau) &= \frac{R(\overline{\mathrm{K}}^0 \rightarrow \mathrm{K}^0)(\tau) - R(\mathrm{K}^0 \rightarrow \overline{\mathrm{K}}^0)(\tau)}{R(\overline{\mathrm{K}}^0 \rightarrow \mathrm{K}^0)(\tau) + R(\mathrm{K}^0 \rightarrow \overline{\mathrm{K}}^0)(\tau)} \\
&= \frac{|\Lambda_{12}|^2 - |\Lambda_{21}|^2}{|\Lambda_{12}|^2 + |\Lambda_{21}|^2} = 4 \cdot \Re e\, \varepsilon_\mathrm{T} + \mathcal{O}(\Delta\varphi^3) && (31)
\end{aligned}$$

and

$$\begin{aligned}
A_{CPT}(\tau) &= \frac{R(\overline{\mathrm{K}}^0 \rightarrow \overline{\mathrm{K}}^0)(\tau) - R(\mathrm{K}^0 \rightarrow \mathrm{K}^0)(\tau)}{R(\overline{\mathrm{K}}^0 \rightarrow \overline{\mathrm{K}}^0)(\tau) + R(\mathrm{K}^0 \rightarrow \mathrm{K}^0)(\tau)} \\
&= -4\Re e\,(\delta) + \mathcal{O}(|\delta|^3) \quad \text{for } \tau \rightarrow \infty && (32)
\end{aligned}$$

In addition by measuring the time dependent rate difference of $\Delta S = 2$ and $\Delta S = 0$ transitions, we obtain a measurement of the oscillation frequency $\Delta\mathrm{m}$

free of any assumptions of symmetry violation in the mass and decay matrix in first order:

$$A_{\Delta \mathrm{m}}(\tau) = \frac{\left[R(\overline{\mathrm{K}}^0 \to \overline{\mathrm{K}}^0)(\tau) + R(\mathrm{K}^0 \to \mathrm{K}^0)(\tau)\right] - \left[R(\overline{\mathrm{K}}^0 \to \mathrm{K}^0)(\tau) + R(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)(\tau)\right]}{\left[R(\overline{\mathrm{K}}^0 \to \overline{\mathrm{K}}^0)(\tau) + R(\mathrm{K}^0 \to \mathrm{K}^0)(\tau)\right] + \left[R(\overline{\mathrm{K}}^0 \to \mathrm{K}^0)(\tau) + R(\mathrm{K}^0 \to \overline{\mathrm{K}}^0)(\tau)\right]}$$

$$= 2\cos(\Delta \mathrm{m}\tau)\frac{\mathrm{e}^{-\overline{\Gamma}\tau}}{\mathrm{e}^{-\Gamma_{\mathrm{S}}\tau} + \mathrm{e}^{-\Gamma_{\mathrm{L}}\tau}} \tag{33}$$

3.5 The $\mathcal{CP}$ violation parameters η_{+-} and η_{00}

The two pion decay modes of neutral kaons can be decomposed into a $\mathrm{I}=0$ and $\mathrm{I}=2$ amplitude. This allows to make use of the fact that the $\mathrm{I}=0$ amplitude is the dominant decay mode of neutral kaons:

$$A_{+-} = \sqrt{\frac{1}{3}}A_2 + \sqrt{\frac{2}{3}}A_0, \qquad A_{00} = \sqrt{\frac{2}{3}}A_2 - \sqrt{\frac{1}{3}}A_0$$
$$A_{0,2} \equiv \sqrt{2\pi}\langle \pi\pi, \mathrm{I}=0,2|T|\mathrm{K}^0\rangle$$

Allowing for $\mathcal{CPT}$ violation in the $\mathrm{I}=0,2$ amplitudes, we can write (use $\mathcal{CP}|\pi\pi\rangle = +|\pi\pi\rangle$) :

$$\begin{aligned} A_{0,2} &= (a_{0,2} + b_{0,2})e^{i\delta_{0,2}} \\ \overline{A}_{0,2} &= (a^*_{0,2} - b^*_{0,2})e^{i\delta_{0,2}}e^{i(\phi_{CP}-\overline{\phi}_T)} \end{aligned} \tag{34}$$

where $a_{0,2}$ are the weak decay amplitudes, $\delta_{0,2}$ the strong interaction phase shifts and $b_{0,2}$ $\mathcal{CPT}$ violating amplitudes.

Assuming only small $\mathcal{CP}$ violation, we can write for any $CP=+1$ final state:

$$\begin{aligned} \eta_f &= \varepsilon_{\mathrm{T}} + \delta + \frac{1}{2}\left(1 - \frac{\overline{A}_f}{A_f}e^{-i\varphi_\Gamma}\right) \\ &\approx \varepsilon_{\mathrm{T}} + \delta + \frac{1}{2}\frac{|A_{\pi\pi}| - |\overline{A}_{\pi\pi}|}{|A_{\pi\pi}|} + \frac{i}{2}\left(\varphi_\Gamma - \arg A^*_{\pi\pi}\overline{A}_{\pi\pi}\right) \end{aligned} \tag{35}$$

In addition to $\mathcal{CP}$ violation occuring in $\mathrm{K}^0 - \overline{\mathrm{K}}^0$ mixing we get two additional terms:

- $\frac{i}{2}\left(\varphi_\Gamma - \arg A^*_{\pi\pi}\overline{A}_{\pi\pi}\right)$, a purely imaginary part from a possible phase difference of the decay amplitude compared to φ_Γ,
- $\frac{1}{2}\frac{|A_{\pi\pi}|-|\overline{A}_{\pi\pi}|}{|A_{\pi\pi}|}$, a real part which can either come directly from $\mathcal{CPT}$ violation of the decay amplitudes or from an interference of two decay amplitudes with a different phase. In the case of only one contributing amplitude, this term is $\mathcal{CPT}$ violating.

For the $\pi^+\pi^-$ and $\pi^0\pi^0$ final states, we obtain by using only terms linear in A_2/A_0 (≈ 0.045):

$$\eta_{+-} = \varepsilon_{\mathrm{T}} + \delta + \varepsilon' + \frac{1}{2}\frac{|A_0| - |\overline{A}_0|}{|A_0|} + \frac{i}{2}\left(\varphi_\Gamma - \arg A_0^* \overline{A}_0\right) \tag{36}$$

$$\eta_{00} = \varepsilon_{\mathrm{T}} + \delta - 2\varepsilon' + \frac{1}{2}\frac{|A_0| - |\overline{A}_0|}{|A_0|} + \frac{i}{2}\left(\varphi_\Gamma - \arg A_0^* \overline{A}_0\right) \tag{37}$$

with

$$\varepsilon' = \frac{1}{2\sqrt{2}}\left(\frac{A_2}{A_0} - \frac{\overline{A}_2}{\overline{A}_0}\right) \tag{38}$$

or using eq.34:

$$\eta_{+-} = \varepsilon_{\mathrm{T}} + \delta + \varepsilon' + \Re e\,\frac{b_2}{a_0} + \frac{i}{2}\left(\varphi_\Gamma - \arg A_0^* \overline{A}_0\right) \tag{39}$$

$$\eta_{00} = \varepsilon_{\mathrm{T}} + \delta - 2\varepsilon' + \Re e\,\frac{b_2}{a_0} + \frac{i}{2}\left(\varphi_\Gamma - \arg A_0^* \overline{A}_0\right) \tag{40}$$

with

$$\varepsilon' = \frac{1}{\sqrt{2}} e^{i(\delta_2 - \delta_o)}\left[i\Im m\left(\frac{a_2}{a_0}\right) + \Re e\left(\frac{a_2}{a_0}\right)\left[\Re e\left(\frac{b_2}{a_2}\right) - \Re e\left(\frac{b_0}{a_0}\right)\right]\right]$$

using that $\Im m\,(a_2/a_0) \ll 1$ and therefore $a_2/a_0 \approx \Re e\,(a_2/a_0)$.

In the limit of a dominating $\mathcal{CPT}$ violation in the $\mathrm{I}=0$ amplitude, e.g. $\frac{\Gamma_{22}-\Gamma_{11}}{\Gamma_{11}} \approx \frac{|\overline{A}_0|^2 - |A_0|^2}{|A_0|^2}$, we can write:

$$\eta_{+-} = \varepsilon_{\mathrm{T}} + \delta_\perp + \varepsilon' + \frac{i}{2}\left(\varphi_\Gamma - \arg A_0^* \overline{A}_0\right) \tag{41}$$

$$\eta_{00} = \varepsilon_{\mathrm{T}} + \delta_\perp - 2\varepsilon' + \frac{i}{2}\left(\varphi_\Gamma - \arg A_0^* \overline{A}_0\right) \tag{42}$$

$$\begin{aligned}
\delta_\perp &\equiv \delta + \Re e\left(\frac{b_0}{a_0}\right) \\
&\approx \delta - \frac{\Gamma_{22} - \Gamma_{11}}{2\Delta\Gamma} \\
&= \frac{i\Delta\Gamma - 2\Delta\mathrm{m}}{4\Delta\mathrm{m}^2 + \Delta\Gamma^2}\left[(M_{22} - M_{11}) + \frac{\Delta\mathrm{m}}{\Delta\Gamma}(\Gamma_{22} - \Gamma_{11})\right]
\end{aligned}$$

3.6 Remark about the $\mathcal{CP}$ violation parameter $\eta_{3\pi}$

Only the interference of the $\mathcal{CP}$ violating amplitude, $A_S(\mathrm{I}=1)$, with the $\mathcal{CP}$ allowed, $A_L(\mathrm{I}=1)$, amplitude, remains after integrating over the entire three-pion phase space. The interference of the $\mathcal{CP}$ allowed, $A_S(\mathrm{I}=2)$, with the $\mathcal{CP}$

violating amplitude, $A_S(\mathrm{I}=1)$ can be safely neglected. Similar to the evaluation of η_{+-} we can write:

$$\eta_{3\pi} = \varepsilon_{\mathrm{T}} - \delta + \frac{1}{4}\frac{\int\int |A_{3\pi}|^2 - |\overline{A}_{3\pi}|^2 dX dY}{\int\int |A_{3\pi}|^2 dX dY} + \frac{i}{2}\left(\varphi_\Gamma - \arg\int\int A^*_{3\pi}\overline{A}_{3\pi} dX dY\right) \quad (43)$$

Since only the $\mathrm{I}=1$ amplitude contributes, and by neglecting $\mathcal{CPT}$ violation in the decay amplitude, we can simply write:

$$\eta_{3\pi} = \varepsilon_{\mathrm{T}} - \delta + \frac{i}{2}\left(\varphi_\Gamma - \arg\int\int A^*_{3\pi}\overline{A}_{3\pi} dX dY\right) \quad (44)$$

Any $\mathcal{CP}$ violation different from the pure mixing alters only the imaginary part of $\eta_{3\pi}$. Such a $\mathcal{CP}$ violation comes from the interference of the 3π decay amplitude with the mixing amplitude.

3.7 Indirect Test of $\mathcal{CPT}$ Invariance

The deviation of the phase of the $\mathcal{CP}$ violation parameter η_{+-} from φ_{SW} is sensitive to $\mathcal{CPT}$ violation. For a precise and a less model dependent analysis, we need to know the phase difference $\varphi_\Gamma - \arg A^*_0\overline{A}_0$, i.e. how much is the phase of the $\mathrm{I}=0$ amplitude different from the total decay amplitude. Based on Eq. 4 and using only the important decay modes:

$$\begin{aligned}\Gamma_{12}\mathrm{e}^{\mathrm{i}\varphi_\Gamma} &= A^*_0\overline{A}_0 + A^*_2\overline{A}_2 + \int d\Omega A^*_{\pi\pi\gamma}\overline{A}_{\pi\pi\gamma} + \sum_{\vec{s}}\int d\Omega[A^*_+\overline{A}_+ + A^*_-\overline{A}_-] \\ &+ \int d\Omega[A^*_{\pi^+\pi^-\pi^0}\overline{A}_{\pi^+\pi^-\pi^0} + A^*_{\pi^0\pi^0\pi^0}\overline{A}_{\pi^0\pi^0\pi^0}]\end{aligned} \quad (45)$$

The $2\pi\gamma$ direct emission amplitude and the $I=2$ amplitude can safely be neglected and the phase of the inner $2\pi\gamma$ bremsstrahlung decay amplitude is almost identical to that of A_0. The important contributions are:

$$\begin{aligned}\sum_{\vec{s}}\int d\Omega[A^*_+\overline{A}_+ + A^*_-\overline{A}_-] &= 4\mathrm{e}^{\mathrm{i}\varphi_\Gamma}\Gamma_L \mathrm{BR}(\mathrm{K_L}\to l^+\pi^-\nu)\cdot x \\ \int d\Omega A^*_{+-0}\overline{A}_{+-0} &= -\frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi_\Gamma}\Gamma_L \mathrm{BR}(\mathrm{K_L}\to\pi^+\pi^-\pi^0)\left(1+2i\Im m\left(\varepsilon_{\mathrm{T}}-\delta-\eta_{+-0}\right)\right) \\ \int d\Omega A^*_{000}\overline{A}_{000} &= -\frac{1}{2}\mathrm{e}^{\mathrm{i}\varphi_\Gamma}\Gamma_L \mathrm{BR}(\mathrm{K_L}\to\pi^0\pi^0\pi^0)\left(1+2i\Im m\left(\varepsilon_{\mathrm{T}}-\delta-\eta_{000}\right)\right)\end{aligned}$$

$$\begin{aligned}\Delta\phi = \varphi_\Gamma - \arg(A^*_0\overline{A}_0) &= \frac{\Gamma_L}{\Gamma_S}\left[8\mathrm{BR}(\mathrm{K_L}\to l^+\pi^-\nu)\cdot\Im m\left(x\right)\right. \\ &\quad -2\mathrm{BR}(\mathrm{K_L}\to\pi^+\pi^-\pi^0)\cdot\Im m\left(\varepsilon_{\mathrm{T}}-\delta-\eta_{+-0}\right) \\ &\quad \left.-2\,\mathrm{BR}(\mathrm{K_L}\to\pi^0\pi^0\pi^0)\cdot\Im m\left(\varepsilon_{\mathrm{T}}-\delta-\eta_{000}\right)\right]\end{aligned}$$

Experimentaly, we have $\delta_2 - \delta_0 + \pi/2 = 45^0 \pm 5^0$ and therefore we can write:

$$\begin{aligned}
\varepsilon' &= \varepsilon'_{||} e^{i\varphi_{SW}} + \varepsilon'_{\perp} e^{i\varphi_{SW}+\pi/2} \\
\rightarrow \varepsilon'_{||} &= |\varepsilon_{\mathrm{T}}| \times \Re e\left(\frac{\varepsilon'}{\varepsilon_{\mathrm{T}}}\right) = (3.4 \pm 1.8) \times 10^{-6} \\
\rightarrow \varepsilon'_{\perp} &= |\varepsilon_{\mathrm{T}}| \times \Im m\left(\frac{\varepsilon'}{\varepsilon_{\mathrm{T}}}\right) \approx \frac{1}{3}|\varepsilon_{\mathrm{T}}|(\varphi_{00} - \varphi_{+-}) \\
& \text{with } \varphi_{00} - \varphi_{+-} = (-0.30^\circ \pm 0.88^\circ) \\
\rightarrow \varepsilon'_{\perp} &= (4 \pm 12) \cdot 10^{-6} \approx \Re e\left(\frac{b_2}{a_0}\right) - \Re e\left(\frac{a_2^*}{a_0^*}\frac{b_0}{a_0}\right)
\end{aligned}$$

The phase difference $\varphi_{00} - \varphi_{+-}$ constraints a possible contribution of the $\mathrm{I} = 2$ amplitude.

We can estimate from the measurement of φ_{+-} to a good precision the $\mathcal{CPT}$ violating contribution:

$$\frac{|\delta_\perp|}{|\eta_{+-}|} \approx (\varphi_{+-} - \varphi_{SW}) - \frac{1}{2}\Delta\phi\frac{\cos(\varphi_{+-})}{|\eta_{+-}|} \tag{46}$$

Assuming no $\mathcal{CPT}$ violation in the decay amplitudes, we can estimate a possible $\overline{\mathrm{K}^0} - \mathrm{K}^0$ mass difference:

$$M_{\overline{\mathrm{K}^0}} - M_{\mathrm{K}^0} = \sqrt{4\Delta\mathrm{m}^2 + \Delta\Gamma^2}\left(|\eta_{+-}|(\varphi_{+-} - \varphi_{SW}) - \frac{1}{2}\Delta\phi\cos(\varphi_{+-})\right) \tag{47}$$

4 Experimental Consequences for $\mathcal{CPT}$ Conservation

Since the 3π channel is dominated by the $I = 1$ amplitude, we can assume $\Im m\,(\eta_{\circ\circ\circ}) \approx \Im m\,(\eta_{+-\circ})$. Using the CPLEAR values for $\Im m\,(x) = (4.8 \pm 4.7) \times 10^{-3}$ and $\Im m\,(\eta_{+-0}) = (-7 \pm 14) \times 10^{-3}$ (obtained by fixing $\Re e\,(\eta_{+-0})$ to $\Re e\,(\eta_{+-})$), we obtain for the phase difference:

$$\frac{1}{2}\frac{\varphi_\Gamma - \arg(A_0^*\overline{A}_0)}{|\eta_{+-}|} = 0.15^\circ \pm 0.24^\circ \tag{48}$$

which has a precision of one order of magnitude better than using the PDG values. Together with the world average value of the phase of the $\mathcal{CP}$ violating parameter η_{+-}, $\varphi_{+-} = 43.82^\circ \pm 0.63^\circ$, we obtain the following limit for a possible $\mathcal{CPT}$ violation:

$$\left|\frac{\delta_\perp}{\eta_{+-}}\right| < 1.9\% \tag{49}$$

This might be translated in a limit of the $\mathrm{K}^0-\overline{\mathrm{K}}^0$ mass difference (assuming no $\mathcal{CPT}$ violation in the decay amplitudes) using:

$$\delta_\perp = \frac{i\Delta\Gamma - 2\Delta\mathrm{m}}{4\Delta\mathrm{m}^2 + \Delta\Gamma^2}\left(M_{\overline{\mathrm{K}}^0} - M_{\mathrm{K}^0}\right) \tag{50}$$

$$\left|\frac{M_{\overline{\mathrm{K}}^0} - M_{\mathrm{K}^0}}{M_{\mathrm{K}}}\right| < 9 \cdot 10^{-19}, \tag{51}$$

which is several orders of magnitude better then existing limits for other particles: but still two orders of magnitude away from a possible effect at the planck scale: $\left|\frac{M_{\overline{\mathrm{K}}^0} - M_{\mathrm{K}^0}}{M_{\mathrm{K}}}\right| \approx \frac{M_{\mathrm{K}}}{M_{Pl}} \approx 10^{-20}$

Some information about CPT violation in the $I = 0$ decay amplitude can be obtained by comparing η_{+-} with the semileptonic charge asymmetry ($\delta_l = 2\Re e\,(\varepsilon_{\mathrm{T}} + \delta)$ and assuming no CPT violation in the semileptonic decay amplitudes for the moment:

$$\begin{aligned}
\Re e\,(itapm) - \frac{1}{2}\delta_l &= \frac{1}{2}\frac{|A_0| - |\overline{A}_0|}{|A_0|} \\
&\approx \Re e\left(\frac{b_0}{a_0}\right) \\
&\approx \frac{\Gamma_{22} - \Gamma_{11}}{2\Delta\Gamma}
\end{aligned}$$

Using the PDG values, $\delta_l = (0.327 \pm 0.012) \times 10^{-2}$ and $\Re e\,(\eta_{+-}) = (2.269 \pm 0.023)\cos(44.3^\circ \pm 0.8^\circ) \times 10^{-3}$, we find for a possible decay width difference:

$$\begin{aligned}
\frac{1}{2}\frac{|A_0| - |\overline{A}_0|}{|A_0|} &= (-0.01 \pm 0.07) \times 10^{-3} \\
(\Gamma_{22} - \Gamma_{11}) &= (-1.5 \pm 0.9) \times 10^{-18}\,\mathrm{GeV/c}
\end{aligned} \tag{52}$$

5 Regeneration

The equation of motion inside some material can be written as:

$$\begin{aligned}
i\frac{\partial}{\partial\tau}|\psi(\tau)\rangle &= i\frac{\partial}{\partial\tau}|\psi(\tau)\rangle_{\mathrm{vac}} + i\frac{\partial}{\partial\tau}|\psi(\tau)\rangle_{\mathrm{nucl}} \\
&= \left[\Lambda - \frac{2\pi N}{m}\begin{pmatrix} f(0) & 0 \\ 0 & \overline{f}(0) \end{pmatrix}\right]|\psi(\tau)\rangle
\end{aligned} \tag{53}$$

By comparing eq. 53 with the eq. 2, we can immediately write down the solutions for the time evolution of a neutral kaon state inside some material:

$$r'_S = -\frac{\Lambda_{22} - \Lambda_{11} + \Delta\lambda' - \frac{2\pi N}{m}[\overline{f}(0) - f(0)]}{2\Lambda_{21}} \tag{54}$$

$$r'_L = -\frac{\Lambda_{22}-\Lambda_{11}-\Delta\lambda' - \frac{2\pi N}{m}[\overline{f}(0)-f(0)]}{2\Lambda_{21}} \quad (55)$$

$$\Delta\lambda' = \sqrt{\left\{\Lambda_{22}-\Lambda_{11}+\frac{2\pi N}{m}[\overline{f}(0)-f(0)]\right\}^2 + 4\Lambda_{21}\Lambda_{12}} \quad (56)$$

The mass and lifetime eigenvalues of $\mathrm{K_S}$ and $\mathrm{K_L}$ change equally in first order of the forward scattering amplitudes.

$$m'_L = m_L - \frac{\pi N}{m}\Re e\,[\overline{f}(0)+f(0)] \quad \Gamma'_L = \Gamma_L - \frac{2\pi N}{m}\Im m\,[\overline{f}(0)+f(0)] \quad (57)$$

$$m'_S = m_S - \frac{\pi N}{m}\Re e\,[\overline{f}(0)+f(0)] \quad \Gamma'_S = \Gamma_S - \frac{2\pi N}{m}\Im m\,[\overline{f}(0)+f(0)] \quad (58)$$

And the time evolution of a $\mathrm{K^0}$ or $\overline{\mathrm{K}}^0$ in matter becomes:

$$|\mathrm{K^0}(\tau)\rangle' = e^{i\frac{\pi N}{m}[\overline{f}(0)+f(0)]\tau}\left[|\mathrm{K^0}(\tau)\rangle - \frac{2\pi \mathrm{N}}{\mathrm{m}\Delta\lambda}[\bar{\mathrm{f}}(0)-\mathrm{f}(0)]\mathrm{f}_-(\tau)|\mathrm{K^0}\rangle\right] \quad (59)$$

$$|\overline{\mathrm{K}}^0(\tau)\rangle' = e^{i\frac{\pi N}{m}[\overline{f}(0)+f(0)]\tau}\left[|\overline{\mathrm{K}}^0(\tau)\rangle + \frac{2\pi \mathrm{N}}{\mathrm{m}\Delta\lambda}[\bar{\mathrm{f}}(0)-\mathrm{f}(0)]\mathrm{f}_-(\tau)|\overline{\mathrm{K}}^0\rangle\right] \quad (60)$$

For an arbitrary inital neutral kaon state we can write following the formalism in section 1:

$$\begin{pmatrix} a(\tau) \\ b(\tau) \end{pmatrix} = T_{\mathrm{reg}}(\tau)\begin{pmatrix} a(0) \\ b(0) \end{pmatrix}$$

$$T_{\mathrm{reg}}(t) = e^{i\frac{\pi N}{m}[\overline{f}(0)+f(0)]t}\left\{T(t) - \frac{2\pi N}{m\Delta\lambda}[\overline{f}(0)-f(0)]f_-(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \quad (61)$$

Consider the standard case of an initial pure $|\mathrm{K_L}\rangle$: (τ counts time after regenerator, $\tau_2 = L/\beta\gamma$ time inside regenerator):

$$\begin{aligned} T(\tau)\cdot T_{\mathrm{reg}}(\tau_2)|\mathrm{K_L}\rangle &= e^{i\frac{\pi N}{m}[\overline{f}(0)+f(0)]\tau_2}\{T(\tau+\tau_2) \\ &\quad -\frac{2\pi N}{m\Delta\lambda}[\overline{f}(0)-f(0)]f_-(\tau_2)T(\tau)\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Bigg\}\begin{pmatrix} r_L \\ 1 \end{pmatrix} \\ &= e^{i\frac{\pi N}{m}[\overline{f}(0)+f(0)]\tau_2}\left\{e^{-i\lambda_L(\tau+\tau_2)}\begin{pmatrix} r_L \\ 1 \end{pmatrix} - \right. \\ &\quad \left. \frac{2\pi N}{m\Delta\lambda}[\overline{f}(0)-f(0)]f_-(\tau_2)T(\tau)\begin{pmatrix} r_L \\ -1 \end{pmatrix}\right\} \end{aligned} \quad (62)$$

To get a more simple expression, assume $\mathcal{CPT}$ conservation, i.e. $\Lambda_{11} = \Lambda_{22}$:

$$T(\tau)\cdot T_{\mathrm{reg}}(\tau_2)|\mathrm{K_L}\rangle = e^{i\frac{\pi N}{m}[\overline{f}(0)+f(0)]\tau_2}\left\{e^{-i\lambda_L(\tau+\tau_2)}\begin{pmatrix} r_L \\ 1 \end{pmatrix} - \right.$$

$$\left.\frac{2\pi N}{m\Delta\lambda}[\overline{f}(0)-f(0)]f_-(\tau_2)\begin{pmatrix} \frac{2\Lambda_{12}}{\Delta\lambda} \\ -1 \end{pmatrix} e^{-i\lambda_S}\right\}$$

$$= e^{-i\tau_2\left(\lambda_L - \frac{\pi N}{m}[\overline{f}(0)+f(0)]\right)} \left\{ \begin{pmatrix} r_L \\ 1 \end{pmatrix} e^{-i\lambda_L \tau} + \rho \begin{pmatrix} r_S \\ 1 \end{pmatrix} e^{-i\lambda_S \tau} \right\} \tag{63}$$

$$\rho = \frac{\pi N}{m\Delta\lambda}[\overline{f}(0) - f(0)]\left[e^{i\Delta\lambda\tau_2} - 1\right]$$
$$= i\frac{\pi N \Lambda_S}{\frac{1}{2} - i\Delta \mathrm{m}\tau_S} \cdot \frac{[\overline{f}(0) - f(0)]}{k}\left[1 - e^{\frac{L}{\Lambda_S}(i\Delta\mathrm{m}\tau_S - \frac{1}{2})}\right]$$

where $m = \frac{k}{\gamma\beta}$, $\Lambda_S = \beta\gamma\tau_S$, $\tau_2 = \frac{L}{\beta\gamma}$ and $\Delta\lambda = \Delta\mathrm{m} + \frac{i}{2}\Delta\Gamma$. A similar calculation yields:

$$T(t) \cdot T_{\mathrm{reg}}(t_2)|\mathrm{K_S}\rangle = e^{-it_2\left(\lambda_S - \frac{\pi N}{m}[\overline{f}(0)+f(0)]\right)} \left\{ \begin{pmatrix} r_S \\ 1 \end{pmatrix} e^{-i\lambda_S t} - \rho \begin{pmatrix} r_L \\ 1 \end{pmatrix} e^{-i\lambda_L t} \right\} \tag{64}$$
$$\tag{65}$$

For evaluating $\mathrm{K_S}$ and $\mathrm{K_L}$ decays in vacuum, it is useful to rotate to the $\mathrm{K_S} - \mathrm{K_L}$ basis using the transformation matrix:

$$T_{LS} = \begin{pmatrix} r_S/N_S & r_L/N_L \\ 1/N_S & 1/N_L \end{pmatrix}^{-1} \tag{66}$$

with

$$N_L = \sqrt{1 + |r_L|^2} \qquad N_S = \sqrt{1 + |r_S|^2} \tag{67}$$
$$|N_L|^2 \approx 2 + 4\Re(\varepsilon_\mathrm{T}) \approx |N_S|^2$$

The decay rates for initial pure $\mathrm{K_S}$ and $\mathrm{K_L}$ into $\pi^+\pi^-$ after the regenerator is then given by:

$$\begin{aligned} R(\mathrm{K_S} \to \pi^+\pi^-) &= |(A_S, A_L) \cdot T_{LS} \cdot T(t) \cdot T_{\mathrm{reg}}(\tau_2)|\mathrm{K_S}\rangle|^2 \\ &\sim e^{-\tau_2\left(\Gamma_S - \frac{\pi N}{m}\Re[\overline{f}(0)+f(0)]\right)} \end{aligned}$$
$$\left[e^{-\tau\Gamma_S} + |\rho|^2|\eta_{+-}|^2 e^{-\tau\Gamma_L} - 2|\rho||\eta_{+-}|e^{-\Delta\Gamma/2\tau}\cos(\Delta\mathrm{m}\tau - \varphi_{+-} - \phi_\rho)\right] \tag{68}$$
$$\begin{aligned} R(\mathrm{K_L} \to \pi^+\pi^-) &= |(A_S, A_L) \cdot T_{LS} \cdot T(\tau) \cdot T_{\mathrm{reg}}(\tau_2)|\mathrm{K_L}\rangle|^2 \\ &\sim e^{-\tau_2\left(\Gamma_L - \frac{\pi N}{m}\Re[\overline{f}(0)+f(0)]\right)} \end{aligned}$$
$$\left[e^{-\tau\Gamma_S} + |\rho|^2|\eta_{+-}|^2 e^{-\tau\Gamma_L} + 2|\rho||\eta_{+-}|e^{-\Delta\Gamma/2\tau}\cos(\Delta\mathrm{m} - \varphi_{+-} - \phi_\rho)\right] \tag{69}$$

The decay rates for initial pure $\mathrm{K^0}$ or $\overline{\mathrm{K}}^0$ after a regenerator where the neutral kaons travel first through vacuum before reaching the regenerator can be calculated similarly:

$$R(\mathrm{K^0} \to \pi\pi) = |(A_S, A_L) \cdot T_{LS} \cdot T(\tau) \cdot T_{\mathrm{reg}}(\tau_2) \cdot T(\tau_1)|\mathrm{K^0}\rangle|^2 \tag{70}$$

Moreover, the decay rate after different regenerators can be easily calculated by multiplying the corresponding matrices:

$$R(\mathrm{K^0} \to \pi\pi) = |(A_S, A_L) \cdot T_{LS} \cdot T(\tau) \cdot \prod_i [T^i_{\mathrm{reg}}(\tau_i)] \cdot T(\tau_0)|\mathrm{K^0}\rangle|^2 \tag{71}$$

6 Test of Quantum Mechanics

Some approaches to quantum gravity suggest (Hawkings) that quantum field theory should be modified in such a way that pure quantum-mechanical states evolve into mixed states, which would necessarily entail a violation of CPT. This possibility may be analyzed using the formulation of open quantum-mechanical systems coupled to an unobserved environment(J.Ellis et al.). In this framework, the observed system is described by a density matrix ρ that obeys a modified quantum Liouville equation

$$\dot{\rho} = i[\rho, H] + \not{\delta} H \rho \tag{72}$$

where H is the usual quantum-mechanical Hamiltonian. The extra term $\propto \not{\delta} H$ would induce a loss of quantum coherence in the observed system, and hence a violation of CPT. Since it is conjectured to arise from quantum-gravitational effects, the magnitude of $\not{\delta} H$ may be at most $\mathcal{O}(m_K^2/M_{Pl})$, where $M_{Pl} = 1.2 \times 10^{19}$ GeV/c is the gravitational mass scale obtained from Newton's constant: $M_{Pl} = G_N^{-\frac{1}{2}}$. An equation of the form (72) is supported by one interpretation of string theory, but could have more general applicability.

In the case of the neutral kaon system, if the conservation of energy and strangeness are assumed, the open-system equation introduces three CPT-violating parameters α, β and γ, which are distinct from K^0–$\overline{\mathrm{K}}^0$ mass and lifetime differences $\Delta\mathrm{m} = m_{\mathrm{K}^0} - m_{\overline{\mathrm{K}}^0}$, $\delta\Gamma = \Gamma_{\mathrm{K}^0} - \Gamma_{\overline{\mathrm{K}}^0}$. The open-system evolution equation (72) is solved perturbatively in the small parameters α, β, γ and ε. A second-order formula for $A_{2\pi}(\tau)$ is derived to:

$$\begin{aligned} A_{2\pi}(\tau) \;=\; & \Big\{ 2|\varepsilon| \cos\phi + 4\widehat{\beta} \sin\phi \cos\phi - 8\widehat{\alpha} \sin\phi \cos\phi (|\varepsilon| \sin\phi - 2\widehat{\beta} \cos^2\phi) \\ & -2\sqrt{|\varepsilon|^2 + 4\widehat{\beta}^2 \cos^2\phi}\; e^{\frac{1}{2}(1/\tau_S - 1/\tau_L)\tau} \left[\cos(\Delta\mathrm{m}\tau - \phi - \delta\phi) + \frac{2\widehat{\alpha}}{\tan\phi} X_\alpha \right] \Big\} \\ & / \left\{ 1 + e^{(1/\tau_S - 1/\tau_L)\tau} \left[\widehat{\gamma} + |\varepsilon|^2 - 4\widehat{\beta}^2 \cos^2\phi - 4\widehat{\beta}|\varepsilon| \sin\phi \right] \right\} \end{aligned} \tag{73}$$

where $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ are scaled variables $\alpha/\Delta\Gamma$, $\beta/\Delta\Gamma$, $\gamma/\Delta\Gamma$, with $\Delta\Gamma = 1/\tau_S - 1/\tau_L$ the decay width difference between $\mathrm{K_S}$ and $\mathrm{K_L}$. Other definitions used are:

$$\begin{aligned} \phi \;&\equiv\; \mathrm{atan}\frac{2\Delta\mathrm{m}}{\Delta\Gamma}\,, \qquad \delta\phi \equiv \mathrm{atan}\left(-\frac{2\widehat{\beta}\cos(\phi)}{|\varepsilon|} \right) \\ X_\alpha \;&\equiv\; \cos\delta\phi \sin(\Delta\mathrm{m}\tau - \phi) - \frac{1}{2}|\Delta\Gamma|\tau \tan\phi \cos(\Delta\mathrm{m}\tau - \phi - \delta\phi) \\ & \quad + \sin\phi \cos(\Delta\mathrm{m}\tau - 2\phi - \delta\phi) \end{aligned}$$

A first-order formula for $A_{\Delta\mathrm{m}}(\tau)$ is given by:

$$A_{\Delta\mathrm{m}}(\tau) = \frac{2e^{-\frac{1}{2}(1/\tau_S + 1/\tau_L)\tau} \left[\cos\Delta\mathrm{m}\tau + \frac{2\widehat{\alpha}}{\tan\phi}(\sin\Delta\mathrm{m}\tau - \Delta\mathrm{m}\tau\cos\Delta\mathrm{m}\tau) \right]}{e^{-\tau/\tau_L}(1 + 2\widehat{\gamma}) + e^{-\tau/\tau_S}(1 - 2\widehat{\gamma})} \tag{74}$$

Since the open quantum mechanic formalism changes the time evolution of states, the best limits on the CPT violation parameter α, β and γ are obtained by comparing measurements at early decay times (CPLEAR) and large decay times ($\mathrm{K_L}$-experiments):

$$|\eta_{+-}|^2 = \widehat{\gamma}^2 + |\varepsilon|^2 \frac{\cos(\phi - 2\delta\phi)}{\cos(\phi)\cos(\delta\phi)^2} , \tag{75}$$

where $|\eta_{+-}| = (2.30 \pm 0.035) \times 10^{-3}$ is the $\mathcal{CP}$-violation parameter measured in $\mathrm{K_L}$ decays to $\pi^+\pi^-$,

$$\delta_L = 2\varepsilon \cdot \cos(\phi) - 4\widehat{\beta}\cos(\phi)\sin(\phi) , \tag{76}$$

and δ_L is the semileptonic asymmetry measured in $\mathrm{K_L}$ decays.

7 Appendix A

Decay rates and asymmetries in the case of a $\Delta S = \Delta Q$ violating interaction:

$$\begin{aligned}
A_+ &= \langle \pi^-(\vec{p}_\pi), l^+(\vec{p}_l, \vec{s}), \nu(\vec{p}_\nu)|H_{weak}|\mathrm{K^0}\rangle \\
\overline{A}_+ &= \langle \pi^-(\vec{p}_\pi), l^+(\vec{p}_l, \vec{s}), \nu(\vec{p}_\nu)|H_{weak}|\overline{\mathrm{K}^0}\rangle \\
A_- &= \langle \pi^+(\vec{p}_\pi), l^-(\vec{p}_l, -\vec{s}), \overline{\nu}(\vec{p}_\nu)|H_{weak}|\mathrm{K^0}\rangle \\
\overline{A}_- &= \langle \pi^+(\vec{p}_\pi), l^-(\vec{p}_l, -\vec{s}), \overline{\nu}(\vec{p}_\nu)|H_{weak}|\overline{\mathrm{K}^0}\rangle
\end{aligned}$$

In the framework of the standard model $A_+ \gg \overline{A}_+$ and $\overline{A}_- \gg A_-$ because of the $\Delta S = \Delta Q$ rule:

$$\overline{x} = \frac{\int \Delta\Omega \sum_{\vec{s}} \mathrm{e}^{-\mathrm{i}\varphi_\mathrm{T}} A_+^* \overline{A}_+}{\int \Delta\Omega \sum_{\vec{s}} |A_+|^2} \qquad x = \frac{\int \Delta\Omega \sum_{\vec{s}} \mathrm{e}^{-\mathrm{i}\varphi_\mathrm{T}} A_-^* \overline{A}_-}{\int \Delta\Omega \sum_{\vec{s}} |A_-|^2}$$

Let us define:

$$B_+ = \int \Delta\Omega \sum_{\vec{s}} |A_+|^2 \qquad B_- = \int \Delta\Omega \sum_{\vec{s}} |\overline{A}_-|^2$$

We measure four rates:

1. $R_+(\tau)$: $\mathrm{K^0}$ at $\tau = 0$ and decay to $l^+\pi^-\nu$ at time t:

$$\begin{aligned}
R_+(\tau) = B_+ \Big\{ &[1 - 4\Re e\,(\delta) + 2\Re e\,(\overline{x})]\,\mathrm{e}^{-\Gamma_\mathrm{S}\tau} + [1 + 4\Re e\,(\delta) - 2\Re e\,(\overline{x})]\,\mathrm{e}^{-\Gamma_\mathrm{L}\tau} \\
&+ 2\mathrm{e}^{-\overline{\Gamma}\tau}\cos\Delta\mathrm{m}\tau + [8\Im m\,(\delta) - 4\Im m\,(\overline{x})]\,\mathrm{e}^{-\overline{\Gamma}\tau}\sin\Delta\mathrm{m}\tau \Big\}
\end{aligned} \tag{77}$$

2. $\overline{R}_-(\tau)$: $\overline{\mathrm{K}^0}$ at $\tau = 0$ and decay to $l^-\pi^+\overline{\nu}$ at time t:

$$\begin{aligned}
\overline{R}_-(\tau) = B_- \Big\{ &[1 + 4\Re e\,(\delta) + 2\Re e\,(x)]\,\mathrm{e}^{-\Gamma_\mathrm{S}\tau} + [1 - 4\Re e\,(\delta) - 2\Re e\,(x)]\,\mathrm{e}^{-\Gamma_\mathrm{L}\tau} \\
&+ 2\mathrm{e}^{-\overline{\Gamma}\tau}\cos\Delta\mathrm{m}\tau - [8\Im m\,(\delta) - 4\Im m\,(x)]\,\mathrm{e}^{-\overline{\Gamma}\tau}\sin\Delta\mathrm{m}\tau \Big\}
\end{aligned} \tag{78}$$

3. $R_-(\tau)$: K^0 at $\tau = 0$ and decay to $l^-\pi^+\overline{\nu}$ at time t:

$$\begin{aligned} R_-(\tau) &= B_- \Big\{ [1 - 4\Re e\,(\varepsilon_{\mathrm{T}}) + 2\Re e\,(x)]\, \mathrm{e}^{-\Gamma_{\mathrm{S}}\tau} + [1 - 4\Re e\,(\varepsilon_{\mathrm{T}}) - 2\Re e\,(x)]\, \mathrm{e}^{-\Gamma_{\mathrm{L}}\tau} \\ &\quad -2\,[1 - 4\Re e\,(\varepsilon_{\mathrm{T}})]\, \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau - 4\Im m\,(x) \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau \Big\} \end{aligned} \tag{79}$$

4. $\overline{R}_+(\tau)$: $\overline{\mathrm{K}}^0$ at $\tau = 0$ and decay to $l^+\pi^-\nu$ at time t:

$$\begin{aligned} \overline{R}_+(\tau) &= B_+ \Big\{ [1 + 4\Re e\,(\varepsilon_{\mathrm{T}}) + 2\Re e\,(\overline{x})]\, \mathrm{e}^{-\Gamma_{\mathrm{S}}\tau} + [1 + 4\Re e\,(\varepsilon_{\mathrm{T}}) - 2\Re e\,(\overline{x})]\, \mathrm{e}^{-\Gamma_{\mathrm{L}}\tau} \\ &\quad -2\,[1 + 4\Re e\,(\varepsilon_{\mathrm{T}})]\, \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau + 4\Im m\,(\overline{x}) \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau \Big\} \end{aligned} \tag{80}$$

In the case of the two decay rates forbidden by the $\Delta S = \Delta Q$ rule ($R_-(\tau)$ and $\overline{R_+}(\tau)$, we neglected the terms proportional to $\int \Delta\Omega \sum_{\vec{s}} |A_-|^2/B_-$ and $\int \Delta\Omega \sum_{\vec{s}} |\overline{A}_+|^2/B_+$, which are only important at short decay times $t \ll 1\tau_{\mathrm{S}}$.

$B_+ \neq B_-$ is a sign of direct $\mathcal{CP}$ violation in the semileptonic decay amplitude. It is difficult to distinguish this effect from differences in the K^0 and $\overline{\mathrm{K}}^0$ decay rates due to different K^0 and $\overline{\mathrm{K}}^0$ tagging efficiencies and for the following we assume $B_+ = B_-$.

$$\begin{aligned} A_{CPT} &= \frac{\overline{R}_-(\tau) - R_+(\tau)}{\overline{R}_-(\tau) + R_+(\tau)} \\ &= \frac{[4\Re e\,(\delta) - 2\Re e\,(x_-)]\, E_-(\tau) + [2\Im m\,(x_+) - 4\Im m\,(\delta)]\, \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau}{E_+(\tau) + 2\Re e\,(x_+) E_-(\tau) + \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau - 2\Im m\,(x_-) \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau} \\ &\approx \frac{4\Re e\,(\delta) E_-(\tau) - 2\Re e\,(x_-) E_-(\tau) + [2\Im m\,(x_+) - 4\Im m\,(\delta)]\, \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau}{E_+(\tau) + \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau} \qquad (81) \\ &= -4\Re e\,(\delta) + 2\Re e\,(x_-) \quad \text{for } t \to \infty \qquad (82) \\ A_T &= \frac{\overline{R}_+(\tau) - R_-(\tau)}{\overline{R}_+(\tau) + R_-(\tau)} \\ &= \frac{4\Re e\,(\varepsilon_{\mathrm{T}}) \left[E_+(\tau) - \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau\right] + 2\Re e\,(x_-) E_-(\tau) + 2\Im m\,(x_+) \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau}{E_+(\tau) + 2\Re e\,(x_+) E_-(\tau) - \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau + 2\Im m\,(x_-) \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau} \\ &\approx 4\Re e\,(\varepsilon_{\mathrm{T}}) + 2 \frac{\Im m\,(x_+) \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau + \Re e\,(x_-) E_-(\tau)}{E_+(\tau) - \mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau} \qquad (83) \\ A_{\Delta \mathrm{m}} &= \frac{\left[R_+(\tau) + \overline{R}_-(\tau)\right] - \left[\overline{R}_+(\tau) + R_-(\tau)\right]}{\left[R_+(\tau) + \overline{R}_-(\tau)\right] + \left[\overline{R}_+(\tau) + R_-(\tau)\right]} \\ &= 2 \frac{\mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau + 2\Im m\,(x_-) \mathrm{e}^{-\overline{\Gamma}\tau} \sin \Delta \mathrm{m}\tau}{[1 + 2\Re e\,(x_+)]\, \mathrm{e}^{-\Gamma_{\mathrm{S}}\tau} + [1 - 2\Re e\,(x_+)]\, \mathrm{e}^{-\Gamma_{\mathrm{L}}\tau}} \\ &\approx 2 \frac{\mathrm{e}^{-\overline{\Gamma}\tau} \cos \Delta \mathrm{m}\tau}{\mathrm{e}^{-\Gamma_{\mathrm{S}}\tau} + \mathrm{e}^{-\Gamma_{\mathrm{L}}\tau}} \left[1 + 2\Im m\,(x_-) \mathrm{e}^{-\overline{\Gamma}\tau} \tan \Delta \mathrm{m}\tau - 2\Re e\,(x_+) \frac{E_-(\tau)}{E_+(\tau)}\right] \qquad (84) \end{aligned}$$

$$
\begin{aligned}
A_2 &= \frac{\left[\overline{R}_+(\tau)+\overline{R}_-(\tau)\right]-\left[R_+(\tau)+R_-(\tau)\right]}{\left[\overline{R}_+(\tau)+\overline{R}_-(\tau)\right]+\left[R_+(\tau)+R_-(\tau)\right]} \\
&= 2\frac{\Re e\,(\varepsilon_{\mathrm{T}})\left[E_+(\tau)-\mathrm{e}^{-\overline{\Gamma}\tau}\cos\Delta\mathrm{m}\tau\right]}{E_+(\tau)+2\Re e\,(x_+)E_-(\tau)}+ \\
&\quad +2\frac{\Re e\,(\delta)E_-(\tau)+\left[\Im m\,(x_+)-\Im m\,(\delta)\right]\mathrm{e}^{-\overline{\Gamma}\tau}\sin\Delta\mathrm{m}\tau}{E_+(\tau)+2\Re e\,(x_+)E_-(\tau)} \\
&\approx \frac{2\Re e\,(\varepsilon_{\mathrm{T}})}{E_+(\tau)}\left[E_+(\tau)-\mathrm{e}^{-\overline{\Gamma}\tau}\cos\Delta\mathrm{m}\tau\right]+ \\
&\quad +\frac{2}{E_+(\tau)}\left(\Re e\,(\delta)E_-(\tau)+\left[\Im m\,(x_+)-\Im m\,(\delta)\right]\mathrm{e}^{-\overline{\Gamma}\tau}\sin\Delta\mathrm{m}\tau\right) \qquad (85) \\
&= 2\Re e\,(\varepsilon_{\mathrm{T}}-\delta) \quad \text{for } t\to\infty \\
\delta_l &= \frac{\left[R_+(\tau)+\overline{R}_+(\tau)\right]-\left[R_-(\tau)+\overline{R}_-(\tau)\right]}{\left[R_+(\tau)+\overline{R}_+(\tau)\right]+\left[R_-(\tau)+\overline{R}_-(\tau)\right]} \\
&= 2\frac{\left[\Re e\,(\varepsilon_{\mathrm{T}})-\Re e\,(\delta)+\Re e\,(x_-)\right]\mathrm{e}^{-\Gamma_{\mathrm{S}}\tau}+\left[\Re e\,(\varepsilon_{\mathrm{T}})+\Re e\,(\delta)-\Re e\,(x_-)\right]\mathrm{e}^{-\Gamma_{\mathrm{L}}\tau}-2\Re e\,(\varepsilon_{\mathrm{T}})\mathrm{e}^{-\overline{\Gamma}\tau}\cos\Delta\mathrm{m}\tau+2\Im m\,(\delta)\mathrm{e}^{-\overline{\Gamma}\tau}\sin\Delta\mathrm{m}\tau}{\left[1+2\Re e\,(x_+)\right]\mathrm{e}^{-\Gamma_{\mathrm{S}}\tau}+\left[1-2\Re e\,(x_+)\right]\mathrm{e}^{-\Gamma_{\mathrm{L}}\tau}} \qquad (86) \\
&= 2\left(\Re e\,(\varepsilon_{\mathrm{T}})+\Re e\,(\delta)-\Re e\,(x_-)\right) \quad \text{for } t\to\infty
\end{aligned}
$$

with $E_\pm(\tau)=\dfrac{\mathrm{e}^{-\Gamma_{\mathrm{S}}\tau}\pm\mathrm{e}^{-\Gamma_{\mathrm{L}}\tau}}{2}$ and $x_\pm=\dfrac{\overline{x}\pm x}{2}$

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