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CPLEAR

How to Calculate the Branching Ratio for the CP Allowed $K_S \to \pi^+\pi^-\pi^0$ Decay

T. Nakada

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Paul Scherrer Institute,Villigen, Switzerland

This report summarises the necessary information in order to calculate the branching ratio for the CP allowed $K_S \to \pi^+\pi^-\pi^0$ decays. It also discusses how to propagate errors and to obtain the upper limit for the branching ratio.

1 Kinematics

1.1 Variables

Let us denote k_μ, p^+_μ, p^-_μ and p^0_μ to be the four momentum of K, π^+, π^- and π^0. Since pions are the decay products of the kaon, we have

$$k_\mu = p^+_\mu + p^-_\mu + p^0_\mu \ .$$

We define three Lorentz invariant variables s_+, s_- and s_0 to be

$$s_i = (k_\mu - p^i_\mu)^2 = m^2_{\rm K} + m_i - 2\, m_{\rm K}\, E^*_i$$

where $m_{\rm K}$ is the kaon mass and m_i is the pion mass. We use $m_\pm$ for charged pions and m_0 for the neutral pion and

$$m_{\rm K} = 497.672,\ m_\pm = 139.56995,\ m_0 = 134.9764 \ \ \text{(PDG 1994)}.$$

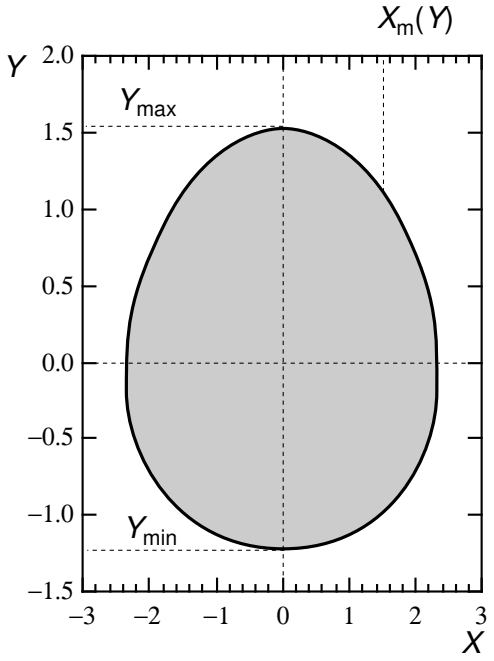

Figure 1: Dalitz plot for the K→ $\pi^+\pi^-\pi^0$ decays. The two kinematical variables X and Y are defined in the text.

The pion energy E_i^* is **measured in the kaon rest frame** so that

$$E_+^* + E_-^* + E_0^* = m_{\rm K} \ .$$

Note that

$$s_+ = m_{-0}^2, \ s_- = m_{+0}^2, \ s_0 = m_{+-}^2$$

where m_{ij} is the invariant mass between the pion-i and -j.

Similarly, we define

$$s_{\rm T} = \frac{s_+ + s_- + s_0}{3} = \frac{m_{\rm K}^2 + 2\,m_\pm^2 + m_0^2}{3} \ .$$

The Dalitz plot for the neutral kaon decays into $\pi^+\pi^-\pi^0$ is often described using two Lorentz invariant kinematical variables, X and Y defined as

$$X = \frac{s_- - s_+}{m_\pm^2} = \frac{2\,m_{\rm K}}{m_\pm^2}\left(E_+^* - E_-^*\right) \quad (1)$$

$$Y = \frac{s_0 - s_{\rm T}}{m_\pm^2} = \frac{2}{3\,m_\pm^2}\left(m_{\rm K}^2 + m_0^2 - m_\pm^2\right) - 2\,\frac{m_{\rm K}}{m_\pm^2}\,E_0^* \ . \quad (2)$$

Note that X becomes $-X$ for the exchange of π^+ and π^- while Y remains the same.

1.2 Kinematical boundaries

Using X and Y, the kinematically allowed region of the Dalitz plot is shown in Figure 1. Note that allowed region of the Dalitz plot is symmetric in X. Here we discuss how to obtain the boundary of the Dalitz plot

Let us calculate the maximum and minimum values of X for a fixed value of Y. From equation 2, the invariant mass of the π^+-π^- subsystem can be written as

$$m_{+-} = \sqrt{m_\pm^2\,Y + s_{\rm T}} \ . \quad (3)$$

Together with equation 1, our problem is reduced to determine the maximum and minimum values of $E_+^* - E_-^*$ for a given value of m_{+-}. The momentum and energy of the π^+-π^- subsystem in the kaon rest frame is given by

$$p_{+-}^* = \frac{\sqrt{\left[m_{\rm K}^2 - (m_{+-} + m_0)^2\right]\,\left[m_{\rm K}^2 - (m_{+-} - m_0)^2\right]}}{2\,m_{\rm K}} \tag{4}$$

and

$$E_{+-}^* = \sqrt{p_{+-}^{*\,2} + m_{+-}^2}\ .$$

The energy and momentum of π^+ in the rest frame of the π^+-π^- subsystem is given by

$$p_+ = \frac{\sqrt{m_{+-}^2 - 4\,m_\pm^2}}{2} \tag{5}$$

and

$$E_+ = \sqrt{p_+^2 + m_\pm^2}\ .$$

The π^+ has the maximum energy in the kaon rest frame when the decay direction of π^+ is parallel to the momentum direction of the π^+-π^- subsystem and given by

$$E_{+\,\rm max}^* = \gamma E_+ + \gamma\,\beta\,p_+$$

where the Lorentz factors are given by

$$\beta = \frac{p_{+-}^*}{E_{+-}^*},\ \gamma = \frac{E_{+-}^*}{m_{+-}} \tag{6}$$

and the corresponding π^- energy in the kaon rest system

$$E_{-\,\rm min}^* = \gamma E_+ - \gamma\,\beta\,p_+\ .$$

Therefore, the maximum value for $E_+^* - E_-^*$ for a given m_{+-} is

$$\left(E_+^* - E_-^*\right)_{\rm max} = 2\,\gamma\,\beta\,p_+ = \frac{2\,p_{+-}^*\,p_+}{m_{+-}}\ . \tag{7}$$

The minimum value is given by

$$\left(E_+^* - E_-^*\right)_{\rm min} = -\,2\,\gamma\,\beta\,p_+\ .$$

The two boundary points for a given m_{+-} are now obtained to be $\pm X_{\rm m}$ where $X_{\rm m}$ is given by

$$X_{\rm m}(Y) = \frac{2\,m_{\rm K}}{m_\pm^2}\ \left(E_+^* - E_-^*\right)_{\rm max}\ .$$

The expression for $\left(E_+^* - E_-^*\right)_{\rm max}$ is derived in equation 7. By combining equations 3, 4, 5 and 6, it follows that

$$X_{\rm m}(Y) = \sqrt{Y^2 - 4\,T\,Y + \left(1 - \frac{4}{T+Y}\right)\,V - \frac{(U-10)\ (5\,U-2)}{9}}$$

where

$$\begin{aligned} T &= \frac{s_{\rm T}}{m_\pm^2} = 5.21662 \\ U &= \frac{m_{\rm K}^2 + m_0^2}{m_\pm^2} = 13.64985 \\ V &= \left(\frac{m_{\rm K}^2 - m_0^2}{m_\pm^2}\right)^2 = 138.75279\ . \end{aligned}$$

At the point where Y (i.e. m_{+-}) is maximum or minimum, we have $X = 0$. From equation 7, it is clear that $X = 0$ can be realised with $p^*_{+-} = 0$ or $p_+ = 0$. As seen from equations 4 and 5, they correspond to $m_{+-} = m_{\rm K} - m_0$ or $m_{+-} = 2\,m_\pm$. Finally we obtain

$$X = 0,\ Y_{\rm min} = \frac{1}{3\,m_\pm^2}\left(10\,m_\pm^2 - m_0^2 - m_{\rm K}^2\right) = -1.21662$$

and

$$X = 0,\ Y_{\rm max} = \frac{2}{3\,m_\pm^2}\left(m_{\rm K}^2 + m_0^2 - m_\pm^2 - 3\,m_{\rm K}\,m_0\right) = 1.53644\ .$$

1.3 Phase space integral

The integrated exclusive decay width is given by

$$\Gamma_{{\rm K}\to\pi^+\pi^-\pi^0} \propto \int\int_{\rm Dalitz\ plot} dX\,dY|A(X,Y)|^2 = \int_{Y_{\rm min}}^{Y_{\rm max}} dY \int_{-X_{\rm m}(Y)}^{X_{\rm m}(Y)} dX\,|A(X,Y)|^2$$

where $A(X,Y)$ is the decay amplitude.

2 Decay Amplitude

We assume that there is no CP violation in the decay amplitude, i.e. all the decay amplitudes are real. The amplitudes for $\rm K_L$ and $\rm K_S$ decaying into $\pi^+\pi^-\pi^0$ can be expressed as

$$A_{\rm L}(X,Y) = \alpha + \beta\,Y + \xi_Y\,Y^2 + \xi_X\,X^2$$

and

$$A_{\rm S}(X,Y) = \epsilon\,A_{\rm L}(X,Y) + \gamma\,X + \xi_{XY}\,XY$$

where α, β, γ and ξ's are real parameters and ϵ is the CP violation parameter in the $\rm K^0$-$\overline{\rm K}^0$ oscillations. It should be noted that the CP violating contribution to the $\rm K_L$ decay is further suppressed due to the centrifugal effect and neglected in $A_{\rm L}$, thus terms proportional to X or $X\,Y$ do not appear[1]. Note also that the contribution to the $\rm K_S$ decay which is CP allowed but suppressed by the centrifugal effect is significantly larger than the CP violating contribution.

The parameters α, β, γ and ξ's are related to the slope parameters in ${\rm K_L} \to \pi^+\pi^-\pi^0$ decays found in PDG, g, h, j and k in the following way:

$$\begin{aligned} |A_{\rm L}(X,Y)|^2 &\approx \alpha^2\left[1 + \frac{2\,\beta}{\alpha}Y + \left(\frac{2\,\xi_Y}{\alpha} + \frac{\beta^2}{\alpha^2}\right)Y^2 + \frac{2\,\xi_X}{\alpha}\,X^2\right] \\ &\propto 1 + g\,Y + h\,Y^2 + j\,X + k\,X^2 \end{aligned}$$

i.e.

$$g = \frac{2\,\beta}{\alpha},\ h = \frac{2\,\xi_Y}{\alpha} + \frac{\beta^2}{\alpha^2},\ j = 0,\ k = \frac{2\,\xi_X}{\alpha}\ .$$

As discussed above, $j = 0$ is a consequence of neglecting the CP violating and kinematically suppressed terms in the $\rm K_L$ decay amplitude.

By measuring the slope parameters, we have a direct access to β, ξ_X and ξ_Y. The current PDG values for the slope parameters are

$$g = 0.670 \pm 0.014,\ h = 0.079 \pm 0.007,\ k = 0.0098 \pm 0.0018\ .$$

From the ${\rm K_S} \to \pi^+\pi^-\pi^0$ decays, one can in principle extract similar slope parameters which measure directly γ and ξ_{XY}. However, this would be possible only with a ϕ factory where $\rm K_S$ can

[1]Detail discussion on the structure of the decay amplitude can be found in; T. Nakada, "Review on CP Violation", AIP Conference Proceedings 302, Lepton and Photon Interactions XVI International Symposium, Ithaca 1993, p.425

Table 1: Values of parameters α, β, ξ_X, ξ_Y, γ and ξ_{XY} obtained from a global study of all the kaon decay data (Experiment) by Devlin and Dickey (DD) and Kambor, Missimer and Wyler (KMW). Also shown are results from chiral perturbation theory by Kambor, Missimer and Wyler (KMW) and by H-Y. Cheng (HYC).

parameters	Experiment		Theory	
	DD	KMW	KMW	HYC
α	84.32 ± 0.43	84.35 ± 0.57	84.2	83.1
β	28.31 ± 0.63	28.11 ± 0.49	28.1	28.4
ξ_Y	-1.62 ± 0.16	-1.32 ± 0.50	-1.96	-1.12
ξ_X	0.29 ± 0.05	0.41 ± 0.17	0.28	0.24
γ	2.51 ± 0.36	2.61 ± 0.27	2.89	-
ξ_{XY}	-	0.28 ± 0.68	0.11	-

be tagged. So far, γ and ξ_{XY} are determined from $\mathrm{K}^\pm \to \pi^\pm\pi^\pm\pi^\mp$ and $\to \pi^\pm\pi^0\pi^0$ decays where the data have less sensitivity on those two parameters. In summary, table 1 shows results of global studies of the kaon decay by Devlin and Dickey (DD) and Kambor, Missimer and Wyler (KMW). It also shows the results of calculations based on chiral perturbation theory done by Kambor, Missimer and Wyler (KMW) and by H-Y. Cheng (HYC).

3 Determination of the $\mathrm{K_S}$ Branching Ratio

In the CPLEAR analysis, we treat events with $X > 0$ (i.e. π^+ energy in the kaon rest frame is larger than π^- energy) and $X < 0$ separately. The $\overline{\mathrm{K}}^0$-K^0 time dependent decay rates asymmetry **after correcting the detector acceptance and K^0-$\overline{\mathrm{K}}^0$ normalisation** is given by

$$a_{+-0}(t) = \begin{cases} 2\,\Re\epsilon - 2\,(\Re\eta_{+-0} + \lambda)\; e^{-\Delta\Gamma\, t/2} \cos\Delta m\, t + 2\Im\eta_{+-0}\, e^{-\Delta\Gamma\, t/2} \sin\Delta m\, t & X > 0 \\ 2\,\Re\epsilon - 2\,(\Re\eta_{+-0} - \lambda)\; e^{-\Delta\Gamma\, t/2} \cos\Delta m\, t + 2\Im\eta_{+-0}\, e^{-\Delta\Gamma\, t/2} \sin\Delta m\, t & X < 0 \end{cases}$$

where $\Delta\Gamma = \Gamma_\mathrm{S} - \Gamma_\mathrm{L}$, $\Delta m = m_\mathrm{L} - m_\mathrm{S}$, η_{+-0} is CP Violation parameter in $\mathrm{K_S} \to \pi^+\pi^-\pi^0$ decays and λ is given by

$$\lambda = \frac{\int_{Y_\mathrm{min}}^{Y_\mathrm{max}} dY \int_0^{X_\mathrm{m}(Y)} dX\; (\gamma\, X + \xi_{XY}\, X\, Y)\; A_\mathrm{L}(X,Y)}{\int_{Y_\mathrm{min}}^{Y_\mathrm{max}} dY \int_0^{X_\mathrm{m}(Y)} dX\, |A_\mathrm{L}(X,Y)|^2} \; .$$

As seen from table 1, the value of the parameter ξ_{XY} is not yet well measured experimentally but clearly smaller than γ. Using the value predicted by chiral perturbation theory $\xi_{XY} = 0.11$, it can be shown that the term $\xi_{XY}\, X\, Y$ contributes less than 1% to λ. It follows that

$$\lambda \approx \gamma\, \frac{\int_{Y_\mathrm{min}}^{Y_\mathrm{max}} dY \int_0^{X_\mathrm{m}(Y)} dX\, X\, A_\mathrm{L}(X,Y)}{\int_{Y_\mathrm{min}}^{Y_\mathrm{max}} dY \int_0^{X_\mathrm{m}(Y)} dX\, |A_\mathrm{L}(X,Y)|^2} \; . \tag{8}$$

Once λ is obtained experimentally, the branching ratio for the CP allowed $\mathrm{K_S} \to \pi^+\pi^-\pi^0$ can be obtained as

$$B(\mathrm{K_S} \to \pi^+\pi^-\pi^0) = B(\mathrm{K_L} \to \pi^+\pi^-\pi^0)\, \frac{\tau_\mathrm{S}}{\tau_\mathrm{L}}\, \frac{\Gamma_\mathrm{S}^{\pi^+\pi^-\pi^0}}{\Gamma_\mathrm{L}^{\pi^+\pi^-\pi^0}} \; .$$

Using the relation

$$\frac{\Gamma_\mathrm{S}^{\pi^+\pi^-\pi^0}}{\Gamma_\mathrm{L}^{\pi^+\pi^-\pi^0}} = \frac{\int_{Y_\mathrm{min}}^{Y_\mathrm{max}} dY \int_{-X_\mathrm{m}(Y)}^{X_\mathrm{m}(Y)} dX\, |\gamma\, X|^2}{\int_{Y_\mathrm{min}}^{Y_\mathrm{max}} dY \int_{-X_\mathrm{m}(Y)}^{X_\mathrm{m}(Y)} dX\, |A_\mathrm{L}(X,Y)|^2}$$

and equation 8, it follows that

$$B(\mathrm{K_S} \to \pi^+\pi^-\pi^0) = B(\mathrm{K_L} \to \pi^+\pi^-\pi^0)\, \frac{\tau_\mathrm{S}}{\tau_\mathrm{L}}\, \mathcal{F}\, \lambda^2$$

where $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F} &= \frac{\left[\int_{Y_{\min}}^{Y_{\max}} dY \int_0^{X_{\rm m}(Y)} dX\,|X|^2\right]\left[\int_{Y_{\min}}^{Y_{\max}} dY \int_0^{X_{\rm m}(Y)} dX\,|A_{\rm L}(X,Y)|^2\right]}{\left[\int_{Y_{\min}}^{Y_{\max}} dY \int_0^{X_{\rm m}(Y)} dX\,X\,A_{\rm L}(X,Y)\right]^2} \\ &= 1.479 \pm 0.022\ .\end{aligned}$$

The values and errors of α, β, ξ_Y and ξ_X obtained by Devlin and Dickey (DD in table 1) are used to calculate $\mathcal{F}$. We assume that those errors are uncorrelated. Finally, we obtain

$$\begin{aligned}B(\mathrm{K_S} \to \pi^+\pi^-\pi^0) &= (0.1238 \pm 0.0022)_{\rm B_L} \times \left[(1.726 \pm 0.014) \times 10^{-3}\right]_\tau \times (1.479 \pm 0.022)_{\mathcal{F}} \times \lambda^2 \\ &= (3.160 \pm 0.076) \times 10^{-4} \times \lambda^2\ . \end{aligned} \quad (9)$$

Equation 9 can be used to determine the branching ratio for the CP allowed $\mathrm{K_S} \to \pi^+\pi^-\pi^0$ decays.

4 Error Propagation for $B(\mathrm{K_S} \to \pi^+\pi^-\pi^0)$

4.1 Method

Let us suppose that the estimation of the parameter λ, λ_0 is determined by maximising the likelihood function $\mathcal{L}(\lambda)$ respect to λ, i.e.

$$\frac{\partial \mathcal{L}(\lambda_0)}{\partial \lambda} = 0\ . \quad (10)$$

As seen from figure 2, the lower and upper bound defining the usual 68% confidence region is given by

$$\ln\left[\mathcal{L}(\lambda_0)\right] - \ln\left[\mathcal{L}(\lambda_+)\right] = \ln\left[\mathcal{L}(\lambda_0)\right] - \ln\left[\mathcal{L}(\lambda_-)\right] = 0.5\ .$$

The parameter estimate can be expressed as

$$\lambda = \lambda_0 \begin{array}{l} +\Delta\lambda^+ \\ -\Delta\lambda^- \end{array}$$

where

$$\Delta\lambda^+ = \lambda_+ - \lambda_0$$

and

$$\Delta\lambda^- = \lambda_0 - \lambda_-$$

i.e. the probability that the true value of λ is in between $\lambda_0 - \Delta\lambda^-$ and $\lambda_0 + \Delta\lambda^+$ is 68%.

If the shape of $\ln\left[\mathcal{L}(\lambda)\right]$ is parabolic around λ_0, the probability distribution for the estimate of λ is very close to Gaussian and we obtain a symmetric error which is identical to the variance of the Gaussian, i.e.

$$\Delta\lambda^+ = \Delta\lambda^- \equiv \sigma\ .$$

In this case, the method of least squares will give the same estimate and errors for λ to those obtained by the maximum likelihood method.

Now we introduce a new parameter R which is a function of λ, $R = r(\lambda)$. Note that the inverse transformation $\lambda = r^{-1}(R)$ must be **uniquely defined**. The likelihood function is maximum at $R = R_0$ where

$$\frac{\partial \mathcal{L}\left[r^{-1}(R)\right]}{\partial R} = 0\ .$$

Using equation 10, it follows that at $R = r(\lambda_0)$

$$\begin{aligned}\frac{\partial \mathcal{L}\left[r^{-1}(R_0)\right]}{\partial R} &= \frac{\partial \mathcal{L}(\lambda_0)}{\partial \lambda}\,\frac{\partial r^{-1}(R_0)}{\partial R}\ . \\ &= 0\end{aligned}$$

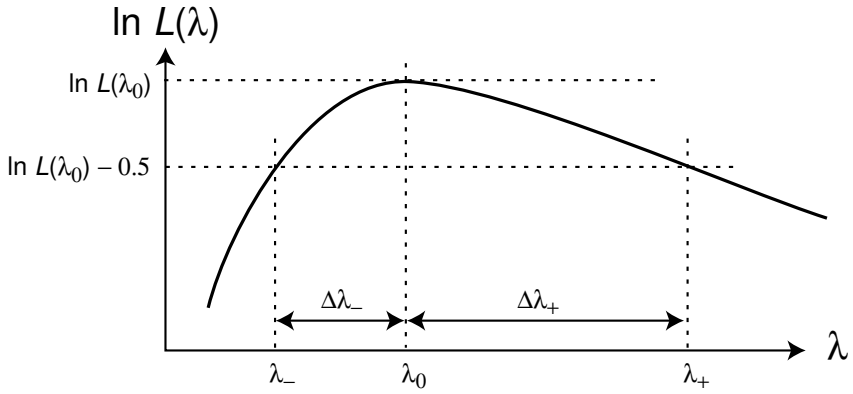

Figure 2: Illustration of the log likelihood function and 68% confidence region.

Thus, we can conclude that the likelihood function is maximum at

$$R_0 = r(\lambda_0) \ . \tag{11}$$

The 68% confidence interval is given similarly by

$$R_- = r(\lambda_-) \leq R \leq R_+ = r(\lambda_+)$$

or

$$R = r(\lambda_0) \ {}^{+\Delta R^+}_{-\Delta R^-}$$

where

$$\Delta R^+ = r(\lambda_+) - r(\lambda_0) \tag{12}$$

and

$$\Delta R^- = r(\lambda_0) - r(\lambda_-) \ . \tag{13}$$

4.2 Application to the August 1994 result

We determine the estimate and errors of the branching ratio for $B(\mathrm{K_S} \to \pi^+\pi^-\pi^0)$ using our current result (August 1994);

$$\begin{aligned} \lambda &= 0.051 \pm 0.016(\text{statistical}) \pm 0.026(\text{systematic}) \\ &\equiv \lambda_0 \pm \sigma_{\text{stat.}} \pm \sigma_{\text{sys.}} \ . \end{aligned} \tag{14}$$

Although the result is obtained by the method of least squares, the χ^2 values around λ_0 is well described by a parabola suggesting that the probability distribution for the estimation of λ is Gaussian. So we assume that we will get the identical results for the estimate and statistical error on λ with the maximum likelihood method.

We would like to know the expectation value and errors on λ^2, i.e.

$$r(\lambda) = \lambda^2 \ .$$

Using equations 11, 12 and 13, the value of estimation for λ^2, ${\lambda^2}_0$, is given by

$${\lambda^2}_0 = (\lambda_0)^2 = (0.051)^2$$

and statistical errors by

$$\Delta\lambda^{2+} = (\lambda_0 + \sigma_{\text{stat.}})^2 - \lambda_0^2 = (0.051 + 0.016)^2 - 0.051^2$$

and

$$\Delta\lambda^{2-} = \lambda_0^2 - (\lambda_0 - \sigma_{\text{stat.}})^2 = 0.051^2 - (0.051 - 0.016)^2$$

Note that we could not apply equations 12 and 13 to obtain the 68% confidence interval if $|\lambda_0| < \sigma$.

By applying the same formula to the systematic errors, we obtain

$$\lambda^2 = 0.0026 \begin{array}{l} +0.0019(\text{statistical}) + 0.0033(\text{systematic}) \\ -0.0014(\text{statistical}) - 0.0020(\text{systematic}) \end{array} \tag{15}$$

Note that if we combine the errors already in equation 14, i.e.

$$\lambda = 0.051 \pm 0.031 \tag{16}$$

it follows that

$$\lambda^2 = 0.0026 \begin{array}{l} +0.0041 \\ -0.0022 \end{array} .$$

I think that this is better than combining the errors in equation 15 .

Using equations 9, 15 and 16 we obtain

$$\begin{aligned} B(\mathrm{K_S} \to \pi^+\pi^-\pi^0) &= 8.2 \begin{array}{l} +6.0(\text{statistical}) + 10.4(\text{systematic}) \\ -4.4(\text{statistical}) - 6.3(\text{systematic}) \end{array} \times 10^{-7} \\ &= 8.2 \begin{array}{l} +13.0 \\ -7.0 \end{array} \times 10^{-7} \end{aligned}$$

5 Upper (and Lower) Limit of $B(\mathrm{K_S} \to \pi^+\pi^-\pi^0)$

5.1 Method

Let us suppose that the likelihood function is maximum at $\lambda = \lambda_0$ and the log-likelihood function is parabolic around its maximum. The a posteriori obtained probability distribution for λ is then given by a Gaussian distribution of

$$P(\lambda) = \frac{1}{\sigma\sqrt{2\,\pi}} \exp\left[\frac{(\lambda - \lambda_0)^2}{2\,\sigma^2}\right]$$

where σ is the parabolic error of λ_0.

The probability distribution for $R = \lambda^2$, $\tilde{P}(R)$ is derived to be

$$\tilde{P}(R) = P\left(\sqrt{R}\right)\, J(\lambda/R) + P\left(-\sqrt{R}\right)\, J(\lambda/R)$$

where the Jacobian $J(\lambda/R)$ is given by

$$\frac{\partial\lambda}{\partial R} = \frac{1}{2\sqrt{R}} .$$

Therefore, the probability distribution for R is given by

$$\tilde{P}(R) = \frac{1}{2\,\sigma\sqrt{2\,\pi\,R}} \left\{ \exp\left[\frac{\left(\sqrt{R} - \lambda_0\right)^2}{2\,\sigma^2}\right] + \exp\left[\frac{\left(-\sqrt{R} - \lambda_0\right)^2}{2\,\sigma^2}\right] \right\} . \tag{17}$$

Note that $\tilde{P}(R)$ is properly normalised, i.e.

$$\int_0^\infty \tilde{P}(R)\, dR = 1$$

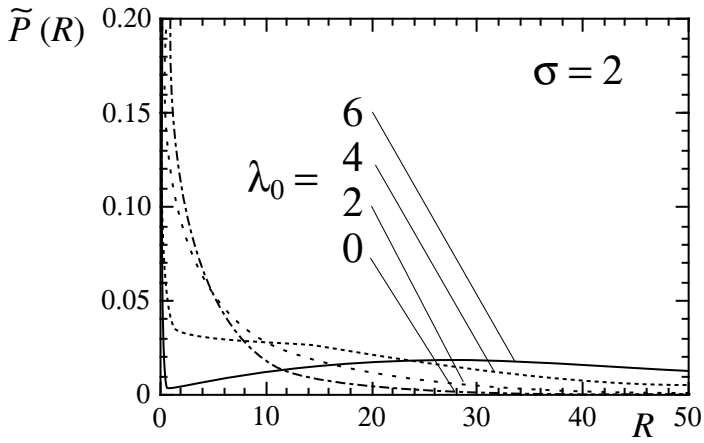

Figure 3: The probability distribution for $R = \lambda^2$ obtained from the probability distribution of λ which is a Gaussian with a variance of $\sigma = 2$ and with four different mean values of $\lambda_0 = 0$, 2, 4 and 6.

Figure 3 shows $\tilde{P}(R)$ for $\lambda_0 = 0$, 2, 4 and 6 for a fixed value of $\sigma = 2$. At $R = 0$, the probability becomes infinite due to the $1/\sqrt{R}$ term in equation 17, however the integral remains finite. The expectation value of R is given by $\lambda_0^2 + \sigma^2$.

The upper limit of R with 90% confidence, $R^{\rm U}_{0.9}$ is given by

$$\int_0^{R^{\rm U}_{0.9}} \tilde{P}(R)\, dR = 0.9 \ . \tag{18}$$

Similarly, the lower limit with 90% confidence, $R^{\rm L}_{0.9}$ is given by

$$\int_{R^{\rm L}_{0.9}}^{\infty} \tilde{P}(R)\, dR = 0.9 \ .$$

5.2 Application to the August 1994 result

If we combine the statistical and systematic errors, the significance of our result on λ given by equation 16 is not very large. Therefore, we might use our result to obtain the upper limit for $B(\mathrm{K_S} \to \pi^+\pi^-\pi^0)$.

Using equations 17 and 18, the upper limit of λ^2 with 90% confidence level is obtained to be 0.0083. Using equation 9, it follows that

$$B(\mathrm{K_S} \to \pi^+\pi^-\pi^0) \leq 2.6 \times 10^{-6} \quad \text{with 90\% confidence} \ .$$

References

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